Math 242, Exam 3, Spring 2017, 1:15 Class

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The exams will be returned in class on Thursday, March 30.

No Calculators or Cell phones.

(1) **State the Existence and Uniqueness Theorem for second order linear Differential Equations.**

Consider the Initial Value Problem

$$y'' + P_1(x)y' + P_2(x)y = Q(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

If P_1, P_2, and Q are continuous on some open interval I which contains x_0, then the Initial Value Problem has a unique solution which is defined on all of I.

(2) **Find the general solution of $\frac{dy}{dx} + \frac{y}{x} = y^2$.**

This is a Bernoulli equation. Let $v = y^{-1}$. Compute $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides by $-y^{-2}$ to obtain $-y^{-2}\frac{dy}{dx} - \frac{y^{-1}}{x} = -1$ or

$$\frac{dv}{dx} - \frac{1}{x}v = -1.$$

Multiply both sides by $e^{\int -\frac{1}{x}dx} = \frac{1}{x}$ to obtain

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = -\frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}v\right) = -\frac{1}{x}$$

$$\frac{1}{x}v = -\ln|x| + C$$

$$v = -x\ln|x| + Cx$$

$$\frac{1}{y} = -x\ln|x| + Cx$$

$$\boxed{\frac{1}{-x\ln|x| + Cx} = y}$$

Check Plug $\frac{1}{-x\ln(x)+Cx} = y$ into the left side of the differential equation to obtain

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{-1-\ln x + C}{(-x\ln(x)+Cx)^2} + \frac{1}{(-x\ln(x)+Cx)}$$

$$= \frac{1}{(-x\ln(x)+Cx)^2}\left[-(-1-\ln x + C) + \frac{(-x\ln(x)+Cx)}{x}\right]$$

$$= \frac{1}{(-x\ln(x)+Cx)^2} = y^2. \checkmark$$

(3) **Find the general solution of $y'' - 4y' + 13y = 0$.**

We consider the characteristic polynomial $r^2 - 4r + 13 = 0$. The roots of this polynomial are

$$r = \frac{4 \pm \sqrt{16 - 4(13)}}{2} = \frac{4 \pm 2\sqrt{4 - (13)}}{2} = 2 \pm 3i.$$

The solution of the differential equation is

$$\boxed{y = e^{2x}(c_1\cos(3x) + c_2\sin(3x)).}$$

Check We compute

$$y' = e^{2x}(-3c_1\sin(3x) + 3c_2\cos(3x) + 2c_1\cos 3x + 2c_2\sin 3x);$$

so

$$y' = e^{2x}([-3c_1 + 2c_2]\sin(3x) + [2c_1 + 3c_2]\cos(3x));$$

$$y'' = e^{2x}([-9c_1+6c_2]\cos(3x)+[-6c_1-9c_2]\sin(3x))+2e^{2x}([-3c_1+2c_2]\sin(3x)+[2c_1+3c_2]\cos(3x)).$$

So,

$$y'' = e^{2x}([-5c_1 + 12c_2]\cos(3x) + [-12c_1 - 5c_2]\sin(3x))$$

Plug the proposed solution into the Differential Equation:

$$y'' - 4y' + 13y = e^{2x}\begin{cases}[-5c_1 + 12c_2]\cos(3x) + [-12c_1 - 5c_2]\sin(3x)) \\ -4([2c_1 + 3c_2]\cos(3x) + [-3c_1 + 2c_2]\sin(3x)) \\ +13(c_1\cos(3x) + c_2\sin(3x))\end{cases}$$

and this is zero. ✓

(4) **Find the general solution of $y'' + y' - 2y = e^x$.**

To solve the homogeneous problem we consider $r^2 + r - 2 = 0$, which is $(r+2)(r-1) = 0$. The general solution of the homogeneous problem is $y = c_1e^x + c_2e^{-2x}$. Now we look for a solution of the non-homogeneous problem. We would try $y = Ae^x$, except this is already a solution of the homogeneous problem. So, instead, we try $y = Axe^x$. We compute $y' = Ae^x(x+1)$ and $y'' = Ae^x(x+2)$. Plug this candidate into the Differential Equation to obtain

$Ae^x(x+2+x+1-2x) = e^x$; so we take $A = 1/3$. The general solution of the Differential Equation is

$$\boxed{y = c_1e^x + c_2e^{-2x} + (1/3)xe^x}.$$

<u>Check:</u> We compute

$$\begin{cases} y = c_1e^x + c_2e^{-2x} + (1/3)xe^x \\ y' = c_1e^x - 2c_2e^{-2x} + (1/3)e^x(x+1) \\ y'' = c_1e^x + 4c_2e^{-2x} + (1/3)e^x(x+2) \end{cases}$$

Plug this into the DE:

$$y'' + y' - 2y = \begin{cases} c_1e^x + 4c_2e^{-2x} + (1/3)e^x(x+2) \\ +c_1e^x - 2c_2e^{-2x} + (1/3)e^x(x+1) \\ -2(c_1e^x + c_2e^{-2x} + (1/3)xe^x) \end{cases}$$

$$= (1/3)e^x(x+2) + (1/3)e^x(x+1) - 2(1/3)xe^x = e^x \checkmark$$

(5) **Solve the initial value problem** $y'' - y = 12e^{3x}$**,** $\quad y(0) = 1$**,** $\quad y'(0) = 9$**.**

The solution of the homogeneous problem is $y = c_1e^x - c_2e^{-x}$. We look for a number A with $y = Ae^{3x}$ is a solution of the given problem; so,

$$Ae^{3x}(9-1) = 12e^{3x}.$$

We take $A = 12/8 = 3/2$. The general solution of the differential equation is

$$y = c_1e^x - c_2e^{-x} + 3/2e^{3x}$$

We compute

$$y' = c_1e^x + c_2e^{-x} + (9/2)e^{3x}$$

$$\begin{cases} 1 = y(0) = c_1 - c_2 + 3/2 \\ 9 = y'(0) = c_1 + c_2 + (9/2) \end{cases}$$

$$\begin{cases} -\frac{1}{2} = c_1 - c_2 \\ \frac{9}{2} = c_1 + c_2 \end{cases}$$

$$\begin{cases} 4 = 2c_1 \\ \frac{9}{2} = c_1 + c_2 \end{cases}$$

$$\begin{cases} 2 = c_1 \\ \frac{5}{2} = c_2 \end{cases}$$

$$\boxed{y = 2e^x - \frac{5}{2}e^{-x} + 3/2e^{3x}}$$

Check: We compute

$$\begin{cases} y = 2e^x - \frac{5}{2}e^{-x} + \frac{3}{2}e^{3x} \\ y' = 2e^x + \frac{5}{2}e^{-x} + \frac{9}{2}e^{3x} \\ y'' = 2e^2x - \frac{5}{2}e^{-x} + \frac{27}{2}e^{3x} \end{cases}$$

$y(0) = 2 - \frac{5}{2} + \frac{3}{2} = 1\checkmark$ $y'(0) = 2 + \frac{5}{2} + \frac{9}{2} = 9\checkmark$

$$y'' - y = (2e^2x - \frac{5}{2}e^{-x} + \frac{27}{2}e^{3x}) - (2e^x - \frac{5}{2}e^{-x} + \frac{3}{2}e^{3x}) = 12e^x.\checkmark$$