Math 242, Exam 3, Fall 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer.

No Calculators or Cell phones.

(1) (12 points) **The acceleration of a car is proportional to the difference between** 250 **ft/sec and the velocity of the car. If this car can accelerate from** 0 **to** 100 **ft/sec in** 10 **seconds, how long will it take for the car to accelerate from rest to** 150 **ft/sec?**

Let $v(t)$ be the velocity of the car (measured in ft/sec) at time t seconds. We are told that $\frac{dv}{dt} = k(250 - v)$. The initial condition is $v(0) = 0$. We are told that $v(10) = 100$. (This allows us to find k.) We are asked to find the time with $v(t) = 150$. We integrate $\int \frac{dv}{250-v} = \int k dt$ to see that

$$-\ln(250 - v) = kt + C \tag{1}$$

The initial condition $v(0) = 0$ tells us that $-\ln 250 = C$. We plug in $v(10) = 100$ into (1) to see that $-\ln(250 - 100) = 10k - \ln 250$. It follows that

$$\ln 250 - \ln(150) = 10k$$

$$\ln \frac{250}{150} = 10k;$$

so, $\frac{\ln \frac{5}{3}}{10} = k$. We now find the time when $v(t) = 150$. Again, we use (1). We solve $-\ln(250 - 150) = kt + C$. We solve $-\ln(100) = (\frac{\ln \frac{5}{3}}{10})t - \ln 250$. We see that $t = \frac{\ln 250 - \ln 100}{\frac{\ln \frac{5}{3}}{10}} = 10\frac{\ln \frac{250}{100}}{\ln \frac{5}{3}} = \boxed{10\frac{\ln \frac{5}{2}}{\ln \frac{5}{3}} \text{sec}}$.

(2) (12 points) **Consider the initial value problem** $\frac{dy}{dx} = x + y^2$, $y(1) = 2$. **Use Euler's method to approximate** $y(12/10)$. **Use two steps, each of size** $1/10$.

Let $f(x, y) = x + y^2$. We consider three points (x_0, y_0), (x_1, y_1), and (x_2, y_2), with $(x_0, y_0) = (1, 2)$, $x_1 = 11/10$, $x_2 = 12/10$, the slope of the line joining (x_0, y_0) and (x_1, y_1) is $f(x_0, y_0)$ and the slope of the line joining (x_1, y_1) and (x_2, y_2) is $f(x_1, y_1)$. The number y_2 is then our approximation of $y(12/10)$.

We first make the slope of the line joining (x_0, y_0) and (x_1, y_1) be $f(x_0, y_0)$:

$$\frac{y_1 - 2}{1/10} = 1 + 2^2;$$

so $y_1 = 2 + \frac{1}{2}$.

We now make the slope of the line joining (x_1, y_1) and (x_2, y_2) be $f(x_1, y_1)$:

$$\frac{y_2 - 5/2}{1/10} = 11/10 + (5/2)^2;$$

Thus,

$$\boxed{\text{our approximation of } y(12/10) \text{ is } y_2 = 5/2 + (11/10 + (5/2)^2)(1/10).}$$

(3) (13 points) **Solve the Initial Value Problem**

$$\begin{cases} y'' - 2y' + y = 2e^x \\ y(0) = 2, \quad y'(0) = 4. \end{cases}$$

Please check your answer.

We first solve the homogeneous problem $y'' - 2y' + y = 0$. We try $y = e^{rx}$ and deal with the characteristic polynomial $r^2 - 2r + 1 = 0$, which is $(r-1)^2 = 0$. The corresponding solutions of the homogeneous differential equation are $y = e^x$ and $y = xe^x$. We would look for a solution of the non-homogeneous problem of the form $y = Ae^x$; however this is a solution of the homogeneous problem; so we would try $y = Axe^x$ instead. Alas, this is also a solution of the homogeneous problem; so we try $y = Ax^2e^x$. We want to find A so that $y = Ax^2e^x$ is a solution of $y'' - 2y' + y = 2e^x$. We take derivatives of $y = Ax^2e^x$:

$$y' = Ax^2e^x + 2Axe^x = Ae^x(x^2 + 2x)$$

and

$$y'' = Ae^x(2x + 2) + Ae^x(x^2 + 2x) = Ae^x(x^2 + 4x + 2).$$

We plug our candidate into the DE. We want to find A with

$$Ae^x(x^2 + 4x + 2) - 2Ae^x(x^2 + 2x) + Ax^2e^x = 2e^x.$$

We want to find A with

$$Ae^x(x^2(1 - 2 + 1) + x(4 - 4) + 2) = 2e^x.$$

We want

$$Ae^x(2) = 2e^x.$$

We take $A = 1$. The general solution of the DE is

$$y = c_1e^x + c_2xe^x + x^2e^x.$$

We use the initial conditions to find c_1 and c_2. We compute

$$y' = e^xc_1 + c_2(xe^x + e^x) + x^2e^x + 2xe^x.$$

$$2 = y(0) = c_1$$

$$4 = y'(0) = c_1 + c_2$$

Thus, $c_1 = 2$ and $c_2 = 2$. The solution of the initial value problem is

$$y = 2e^x + 2xe^x + x^2e^x;$$

which is the same as

$$y = e^x(2 + 2x + x^2).$$

Check. We compute

$$y' = e^x(2 + 2x) + e^x(2 + 2x + x^2) = e^x(4 + 4x + x^2)$$

$$y'' = e^x(4 + 2x) + e^x(4 + 4x + x^2) = e^x(8 + 6x + x^2).$$

We see that $y(0) = 2$, $y'(0) = 4$, and when we plug y into the left side of the DE we get

$$e^x(8 + 6x + x^2) - 2e^x(4 + 4x + x^2) + e^x(2 + 2x + x^2),$$

and this is $2e^x$, as expected.

(4) (13 points) **Find the general solution of $xy' + 4y = x^3$. Please check your answer.**

This is a first order linear DE. Divide by x:

$$y' + \frac{4}{x}y = x^2.$$

Multiply both sides by $\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$:

$$x^4y' + 4x^3y = x^6.$$

Integrate both sides:

$$x^4y = \frac{x^7}{7} + C.$$

$$\boxed{y = \frac{x^3}{7} + Cx^{-4}.}$$

Check. Plug y into the left side of the DE to get

$$\frac{3x^2}{7} - 4Cx^{-5} + \left(\frac{4}{x}\right)\left(\frac{x^3}{7} + Cx^{-4}\right)$$

$$= \frac{3x^2}{7} + \frac{4x^2}{7} + C(-4x^{-5} + 4x^{-5}) = x^2 \checkmark.$$