Math 242, Exam 2, Fall, 2023 Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Solve** $x\frac{dy}{dx} + 6y = 3xy^{4/3}$**.**

This is a Bernoulli equation. Let $v = y^{1-4/3} = y^{-1/3}$. Compute

$$\tfrac{dv}{dx} = \tfrac{-1}{3}y^{-4/3}\tfrac{dy}{dx}.$$

Multiply both sides of the equation by $\frac{-1}{3}y^{-4/3}$ to obtain

$$x(\tfrac{-1}{3})y^{-4/3}\tfrac{dy}{dx} - 2y^{-1/3} = -x$$

$$x\tfrac{dv}{dx} - 2v = x.$$

Divide both sides by x to obtain

$$\tfrac{dv}{dx} - \tfrac{2}{x}v = -1. \quad (*)$$

Let

$$\mu = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}.$$

Multiply both sides of (*) by x^{-2}

$$x^{-2}\tfrac{dv}{dx} - 2x^{-3}v = -x^{-2}.$$

The left side is $\frac{d}{dx}(x^{-2}v)$. So the equation is

$$\tfrac{d}{dx}(x^{-2}v) = -x^{-2}.$$

Integrate both sides to obtain

$$x^{-2}v = x^{-1} + C.$$

Multiply both sides by x^2 to get

$$v = x + Cx^2.$$

$$y^{-1/3} = Cx^2 + x$$

$$\boxed{y = (Cx^2 + x)^{-3}.}$$

(2) **Solve the Initial Value Problem**

$$\tfrac{dy}{dx} = -6xy, \quad y(0) = 7.$$

Separate the variables and integrate

$$\int \frac{dy}{y} = -\int 6x\,dx$$
$$\ln|y| = -3x^2 + C$$
$$|y| = e^C e^{-3x^2}$$
$$y = \pm e^C e^{-3x^2}$$

Let K be the constant $\pm e^C$.

$$y = Ke^{-3x^2}.$$
$$7 = y(0) = Ke^0 = K$$
$$\boxed{y = 7e^{-3x^2}}$$

(3) **Solve the Initial Value problem**

$$\tfrac{dx}{dt} = 3 - x, \quad x(0) = x_0.$$

Graph the solution of the Initial Value Problem for a few different choices of x_0.

Separate the variables and integrate:

$$\frac{dx}{3-x} = dt$$
$$-\ln|3-x| = t + C$$
$$\ln|3-x| = -t - C$$
$$|3-x| = e^{-C}e^{-t}$$
$$3 - x = \pm e^{-C}e^{-t}$$
$$3 - \pm e^{-C}e^{-t} = x$$

Let K be the constant $\pm e^{-C}$.

$$3 - Ke^{-t} = x$$
$$3 - K = x_0;$$

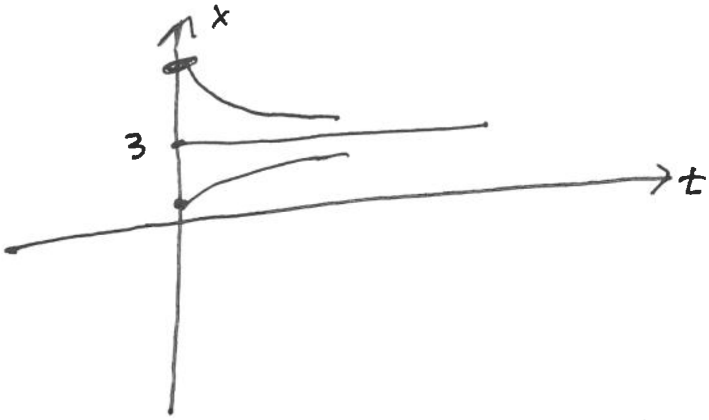
so $3 - x_0 = K$. We conclude that

$$x(t) = 3 - (3 - x_0)e^{-t}$$

or

$$\boxed{x(t) = 3 + (x_0 - 3)e^{-t}.}$$

If $x_0 = 3$, the $x(t) = 3$ for all t. If $3 < x_0$, then $x(t)$ is always decreasing toward 3. If $x_0 < 3$, then $x(t)$ is always growing toward 3.



some solutions of $\frac{dx}{dt} = 3 - x$

$x(0) = x_0$

(4) **Find the general solution of $y^{(4)} - 8y^{(3)} + 16y'' = 0$.(In this problem, y is a function of x.)**

Try $y = e^r x$; study the characteristic equation

$$r^4 - 8r^3 + 16r^2 = 0$$

$$r^2(r-4)^2 = 0$$

The roots are $r = 0$ with multiplicity two and $r = 4$ with multiplicity two. The general solution of the DE is

$$\boxed{y = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}.}$$

(5) **Find the general solution of $y'' + 4y = 3x^3$. (In this problem, y is a function of x.)**

We first solve the homogeneous problem $y'' + 4y = 0$. Try $y = e^r x$. The characteristic equation is $r^2 + 4 = 0$. The roots are $\pm 2i$. The general solution of the homogeneous problem is $y = c_1 \cos 2x + c_2 \sin 2x$. Now we look for a particular solution of the given problem of the form $y = Ax^3 + Bx^2 + Cx + D$. We compute $y' = 3Ax^2 + 2Bx + C$ and $y'' = 6Ax + 2B$. Plug this candidate into the DE

$$(6Ax + 2B) + 4(Ax^3 + Bx^2 + Cx + D) = 3x^3$$

Solve

$$\begin{cases} 4A = 3 \\ 4B = 0 \\ 6A + 4C = 0 \\ 2B + 4D = 0 \end{cases}$$

Conclude that $A = \frac{3}{4}$, $B = 0$, $C = -\frac{9}{8}$, $D = 0$.

The general solution of the given Differential Equation is

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{3}{4}x^3 - \frac{9}{8}x.}$$