Math 242, Exam 2, Spring 2017, 1:15 Class solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The exams will be returned in class on Tuesday, Feb. 28.

No Calculators or Cell phones.

(1) (a) **State the Existence and Uniqueness Theorem for second order linear Differential Equations.**

Consider the Initial Value Problem

$$y'' + P_1(x)y' + P_2(x)y = Q(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

If P_1, P_2, and Q are continuous on some open interval I which contains x_0, then the Initial Value Problem has a unique solution which is defined on all of I.

(b) **What does what does (a) tell you about the Initial Value Problem**

$$e^x \frac{d^2y}{dx^2} + \frac{1}{x-3}\frac{dy}{dx} = x, \quad y(1) = -1, \quad y'(1) = 6?$$

(Please explain your answer.)

Our initial value problem is

$$\frac{d^2y}{dx^2} + \frac{1}{e^x(x-3)}\frac{dy}{dx} = \frac{x}{e^x}, \quad y(1) = -1, \quad y'(1) = 6?$$

So $P_1(x) = \frac{1}{e^x(x-3)}$, $P_2(x) = 0$, and $Q(x) = \frac{x}{e^x}$. Let I be the open interval $(-\infty, 3)$ (or $\{x \in \mathbb{R} \mid x < 3\}$). The function $P_1(x)$ is not defined at $x = 3$; but $P_1(x)$, $P_2(x)$, and $Q(x)$ all are continuous on I. The Theorem guarantees that the initial value problem has a unique solution and this solution is defined on all of I.

(2) **A tank with 200 gallons of brine solution contains 40 lbs of salt. A brine solution with 2 pounds of salt per gallon of solution is pumped into the tank at a rate of 4 gal/min. The well mixed solution is pumped out of the tank out at a rate of 4 gal/min. How much salt is in the tank after 1 hour? How much salt is in the tank after a very long time? SET UP THE INITIAL VALUE PROBLEM. DO NOT SOLVE THE INITIAL VALUE PROBLEM.**

Let $x(t)$ be the number of pounds of salt in the tank at time t, where t is measured in minutes. We are told that $x(0) = 40$. The rate of change of x is the rate in minus the rate out. The rate in is

$$\frac{2 \text{ pounds}}{\text{gal}} \frac{4 \text{ gal}}{\text{min}}.$$

The rate out is

$$\frac{x(t) \text{ pounds}}{200 \text{ gal}} \frac{4 \text{ gal}}{\text{min}}.$$

The Initial Value Problem is

$$\boxed{\frac{dx}{dt} = 8 - \frac{x}{50}, \quad x(0) = 40.}$$

(3) **Solve $xy\frac{dy}{dx} + 4x^2 + y^2 = 0$. Express your answer in the form $y = y(x)$. Please check your answer.**

We make a homogeneous substitution. Let $v = \frac{y}{x}$. We compute $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. Divide both sides by x^2:

$$\frac{y}{x}\frac{dy}{dx} + 4 + (\tfrac{y}{x})^2 = 0$$

$$v(x\tfrac{dv}{dx} + v) + 4 + v^2 = 0$$

$$xv\tfrac{dv}{dx} + 4 + 2v^2 = 0$$

$$xv\tfrac{dv}{dx} = -(4 + 2v^2)$$

$$\int \tfrac{v}{(4+2v^2)} dv = -\int \tfrac{1}{x} dx$$

$$\tfrac{1}{4}\ln(4 + 2v^2) = -\ln|x| + C$$

$$\ln(4 + 2v^2) = -4\ln|x| + 4C$$

$$4 + 2v^2 = e^{4C}e^{-4\ln|x|}$$

Let K represent the constant e^{4C}.

$$4 + 2v^2 = K\tfrac{1}{x^4}$$

$$2v^2 = K\tfrac{1}{x^4} - 4$$

$$(\tfrac{y}{x})^2 = \tfrac{K\frac{1}{x^4} - 4}{2}$$

$$y^2 = x^2\tfrac{K\frac{1}{x^4} - 4}{2}$$

$$y^2 = \tfrac{\frac{K}{x^2} - 4x^2}{2}$$

$$\boxed{y = \pm\sqrt{\tfrac{\frac{K}{x^2} - 4x^2}{2}}}$$

Check. We plug our proposed solution ($y = +\sqrt{\frac{\frac{K}{x^2}-4x^2}{2}}$) into the left side of the given differential equation:

$$xy\frac{dy}{dx} + 4x^2 + y^2 = x\sqrt{\frac{\frac{K}{x^2} - 4x^2}{2}}\,\frac{\frac{1}{2}(\frac{-2K}{x^3} - 8x)}{2\sqrt{\frac{\frac{K}{x^2}-4x^2}{2}}} + 4x^2 + \frac{\frac{K}{x^2} - 4x^2}{2}$$

$$= x\frac{\frac{1}{2}(\frac{-2K}{x^3} - 8x)}{2} + 4x^2 + \frac{\frac{K}{x^2} - 4x^2}{2}$$

$$= \frac{(\frac{-K}{x^2} - 4x^2)}{2} + 4x^2 + \frac{\frac{K}{x^2} - 4x^2}{2}$$

$$= -4x^2 + 4x^2 = 0.\checkmark$$

(4) **Solve** $\frac{dy}{dx} = (y-1)(y-3)$**. Draw some of the solutions of this Differential Equation for various values of** $y(0)$**.**

The picture is drawn elsewhere. We see that $y(x) = 1$ and $y(x) = 3$ are equilibrium solutions of the differential equation. Other solutions are attracted to $y = 1$ and flee $y = 3$.

We separate the variables and integrate.

$$\int \frac{dy}{(y-1)(y-3)} = \int dx.$$

We look for constants A and B with

$$\frac{A}{y-1} + \frac{B}{y-3} = \frac{1}{(y-1)(y-3)}.$$

Multiply both sides by $(y-1)(y-3)$ to obtain

$$A(y-3) + B(y-1) = 1.$$

Plug in $y = 3$ to learn $B = \frac{1}{2}$. Plug in $y = 1$ to learn that $A = \frac{-1}{2}$. We verify that

$$\frac{1}{2}\left[\frac{-1}{y-1} + \frac{1}{y-3}\right] = \frac{1}{(y-1)(y-3)}.$$

The left side is

$$\frac{1}{2}\left[\frac{-(y-3)+(y-1)}{(y-1)(y-3)}\right] = \frac{1}{2}\left[\frac{2}{(y-1)(y-3)}\right] = \frac{1}{(y-1)(y-3)}.$$

We integrate

$$\frac{1}{2}\int\left[\frac{-1}{y-1} + \frac{1}{y-3}\right]dy = \int dx$$

to obtain

$$\frac{1}{2}(\ln|y-3| - \ln|y-1|) = x + C.$$

$$\ln\left|\frac{y-3}{y-1}\right| = 2x + 2C.$$

$$\left|\frac{y-3}{y-1}\right| = e^{2C}e^{2x}$$
$$\frac{y-3}{y-1} = \pm e^{2C}e^{2x}$$

Let K represent the constant $\pm e^{2c}$.

$$\frac{y-3}{y-1} = Ke^{2x}$$

We notice that
$$\frac{y(0)-3}{y(0)-1} = K.$$
$$y - 3 = Ke^{2x}(y-1)$$
$$y(1 - Ke^{2x} = 3 - Ke^{2x}$$
$$\boxed{y(x) = \frac{3-Ke^{2x}}{1-Ke^{2x}}, \text{ with } K = \frac{y(0)-3}{y(0)-1}.}$$

(5) **Find all constants r for which $y = e^{rx}$ a solution of $y'' + 3y' + 2y = 0$. Find a constant A with $y = Ae^{3x}$ a solution of $y'' + 3y' + 2y = e^{3x}$. What is the general solution of $y'' + 3y' + 2y = e^{3x}$?**

We plug e^{rx} into $y'' + 3y' + 2y = 0$ to obtain

$$r^2e^{rx} + 3re^{rx} + 2e^{rx} = 0.$$

So,
$$e^{rx}(r^2 + 3r + 2) = 0;$$

but e^{rx} is never zero; so
$$(r^2 + 3r + 2) = 0$$

or
$$(r+2)(r+1) = 0.$$

We have shown that if $y = e^{rx}$ is a solution of $y'' + 3y' + 2y = 0$, then $r = -1$ or $r = -2$ and $y = e^{-x}$ or $y = e^{-2x}$.

If $y = Ae^{3x}$ a solution of $y'' + 3y' + 2y = e^{3x}$, then

$$9Ae^{3x} + 9Ae^{3x} + 2Ae^{3x} = e^{3x};$$

hence
$$20Ae^{3x} = e^{3x}$$

and $A = \frac{1}{20}$.

We have found two linearly independent solutions of the homogeneous problem and one particular solution of the non-homogeneous problem. It follows that the general solution of $y'' + 3y' + 2y = e^{3x}$ is

$$\boxed{y = c_1e^{-x} + c_2e^{-2x} + \tfrac{1}{20}e^{3x}.}$$