Math 242, Exam 2, Fall 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators or Cell phones.

(1) **Solve** $x\dfrac{dy}{dx} + 6y = 3xy^{4/3}$**. Write your answer in the form** $y = y(x)$**. Check your answer.**

This is a Bernoulli equation. Let $v = y^{1-4/3} = y^{-1/3}$. Observe that $\frac{dv}{dx} = \frac{-1}{3}y^{-4/3}\frac{dy}{dx}$. Multiply both sides of the original equation by $\frac{-1}{3}y^{-4/3}$.

$$\frac{-1}{3}y^{-4/3}x\frac{dy}{dx} + \frac{-1}{3}y^{-4/3}6y = 3xy^{4/3}\frac{-1}{3}y^{-4/3}$$

$$x\frac{dv}{dx} - 2v = -x$$

Divide both sides by x:

$$\frac{dv}{dx} - \frac{2}{x}v = -1.$$

Multiply both sides by $\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$:

$$x^{-2}\frac{dv}{dx} - 2x^{-3}v = -x^{-2}.$$

Apply $\int - - \, dx$:

$$x^{-2}v = x^{-1} + C.$$

$$v = x + Cx^2.$$

$$y^{-1/3} = x + Cx^2$$

$$\boxed{y = (x + Cx^2)^{-3}.}$$

<u>Check:</u> Plug $y = (x + Cx^2)^{-3}$ in to $x\frac{dy}{dx} + 6y$ to get

$$x(-3)(x + Cx^2)^{-4}(1 + 2Cx) + 6(x + Cx^2)^{-3}$$

$$= (x + Cx^2)^{-4}[-3x(1 + 2Cx) + 6(x + Cx^2)]$$

$$= (x + Cx^2)^{-4}[-3x + 6(x)]$$

$$= 3x(x + Cx^2)^{-4} = 3xy^{4/3}. \checkmark$$

(2) **Solve** $\frac{dx}{dt} = 7x - x^2 - 10$**. Sketch a few solutions.**

We separate the variables and integrate both sides of $\int \frac{dx}{7x-x^2-10} = \int dt$ or $\int \frac{-dx}{x^2-7x+10} = \int dt$ or $\int \frac{dx}{(x-5)(x-2)} = \int -dt$ Apply the method of partial fractions and look for numbers A and B with

$$\frac{1}{(x-5)(x-2)} = \frac{A}{(x-5)} + \frac{B}{(x-2)}.$$

Multiply both sides by $(x-5)(x-2)$ to obtain

$$1 = A(x-2) + B(x-5).$$

Plug in $x = 5$ to learn $\frac{1}{3} = A$. Plug in $x = 2$ to learn $\frac{-1}{3} = B$. Integrate both sides of

$$\frac{1}{3}\int \left(\frac{1}{(x-5)} - \frac{1}{(x-2)}\right) dx = \int -dt$$

or

$$\int \left(\frac{1}{(x-5)} - \frac{1}{(x-2)}\right) dx = \int -3dt.$$

We obtain

$$\ln|x-5| - \ln|x-2| = -3t + C$$

$$\ln\left|\frac{x-5}{x-2}\right| = -3t + C$$

Exponentiate to obtain

$$\left|\frac{x-5}{x-2}\right| = e^C e^{-3t}$$

$$\frac{x-5}{x-2} = \pm e^C e^{-3t}$$

Let $K = \pm e^C$.

$$\frac{x-5}{x-2} = Ke^{-3t}$$

This might be a good time to notice that

$$\frac{x(0)-5}{x(0)-2} = K.$$

Multiply both sides by $x-2$:

$$x - 5 = (x-2)Ke^{-3t}$$

$$x(1 - Ke^{-et}) = 5 - 2Ke^{-3t}$$

$$x = \frac{5 - 2Ke^{-3t}}{(1 - Ke^{-3t})}.$$

$$x = \frac{5 - 2\frac{x(0)-5}{x(0)-2}e^{-3t}}{1 - \frac{x(0)-5}{x(0)-2}e^{-3t}}.$$

$$\boxed{x(t) = \frac{5(x(0)-2) - 2(x(0)-5)e^{-3t}}{(x(0)-2) - (x(0)-5)e^{-3t}}.}$$

Our solution says that if $x(0) = 2$, then $x(t) = 2$ for all t; similarly if $x(0) = 5$, then $x(t) = 5$ for all t. Of course, we knew that all along, because the original problem was

$$\frac{dx}{dt} = -(x-5)(x-2).$$

The original problem shows us that if $5 < x(0)$, then $\frac{dx}{dt} < 0$ for all t and the solution heads toward the equilibrium $x(t) = 5$. If $2 < x(0) < 5$, then $0 < \frac{dx}{dt}$ and the solution heads toward the equilibrium $x(t) = 5$. If $x(0) < 2$, then $\frac{dx}{dt} < 0$ for all t and the solution flees from the equilibrium solution $x(0) = 2$. One can verify that our solution reflects these properties. At any rate, the sketch we draw (see the other page) does reflect these properties.

(3) **A motor boat is moving at 40 feet per second when its motor suddenly quits and 10 seconds later the boat has slowed to 20 feet/second. The only force acting on the boat is resistance and resistance is proportional to velocity. How far will the boat coast in all?**

Let $x(t)$ equal the position of the boat at time t, where t measures the amount of time since the motor quit. We are told

$$x'' = -kx', \quad x'(0) = 40, \quad x'(10) = 20, \quad \text{and} \quad x(0) = 0,$$

for some positive constant k. If you like, let $v = x'$. Separate the variables in $\frac{dv}{dt} = -kv$ and integrate $\int \frac{dv}{v} = \int -kdt$:

$$\ln|v| = -kt + C$$
$$|v| = e^{-kt+C}$$
$$v = Ke^{-kt}$$
$$x' = Ke^{-kt}$$

Plug in $t = 0$ to learn $K = 40$. So,

$$x' = 40e^{-kt}$$

Plug in $t = 10$ to learn

$$20 = 40e^{-10k}$$
$$\frac{1}{2} = e^{-10k}$$
$$-\ln 2 = -10k$$
$$\frac{\ln 2}{10} = k$$

Integrate with respect to t to see that

$$x = \frac{40}{-k}e^{-kt} + C_1$$

Plug in $t = 0$ to see that

$$0 = \frac{40}{-k} + C_1$$

so $C_1 = \frac{40}{k}$ and $x(t) = \frac{40}{k} - \frac{40}{-k}e^{-kt}$. We see that x' is never zero; but $\lim_{t\to\infty} x' = 0$. The total distance traveled by the boat is

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{40}{k} - \frac{40}{-k}e^{-kt} = \frac{40}{k} = \frac{40}{\frac{\ln 2}{10}} = \boxed{\frac{400}{\ln 2}\text{feet}}.$$

(4) **A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1+\cos t)$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank at time t? Set up the initial value problem. You do not have to solve it.**

Let $x(t)$ equal the number of pounds of salt in the tank at time t. We are told that $x(0) = 5$. We are also told that

$$\frac{dx}{dt} = \frac{1}{5}(1+\cos t)\frac{\text{lbs}}{\text{gal}}9\frac{\text{gal}}{\text{hr}} - \frac{x}{600+3t}\frac{\text{lbs}}{\text{gal}}6\frac{\text{gal}}{\text{hr}}.$$

We must solve the initial value problem:

$$\boxed{\begin{cases} \frac{dx}{dt} = \frac{9}{5}(1+\cos t) - \frac{6x}{600+3t} \\ x(0) = 5. \end{cases}}$$

(5) **Consider the differential equation**

$$y'' - 5y' + 6y = e^{-x}. \tag{1}$$

(a) **Which of the functions $y_1 = e^{-x}$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ is a solution of the corresponding homogeneous problem**

$$y'' - 5y' + 6y = 0?$$

(b) **Find a constant α and a function y_i (selcted from $y_1 = e^{-x}$, $y_2 = e^{2x}$, $y_3 = e^{3x}$) so that $y = \alpha y_i$ is a solution of the original differential equation (1).**

(c) **What is the general solution of the original differential equation (1)?**

(a) We see that $y_2' = 2e^{2x}$ and $y_2'' = 4e^{2x}$. When y_2 is plugged in to the left side of the homogeneous problem we obtain

$$4e^{2x} - 5\cdot 2e^{2x} + 6e^{2x} = (4 - 10 + 6)e^{2x} = 0.$$

$$\boxed{\text{Thus } y_2 \text{ is a solution of the homogeneous problem.}}$$

Similarly, we see that $y_3' = 3e^{3x}$ and $y_3'' = 9e^{3x}$. When y_3 is plugged in to the left side of the homogeneous problem we obtain

$$9e^{3x} - 5 \cdot 3e^{3x} + 6e^{3x} = (9 - 15 + 6)e^{3x} = 0.$$

$$\boxed{\text{Thus } y_3 \text{ is a solution of the homogeneous problem.}}$$

On the other hand, $y_1' = -e^{-x}$ and $y_1'' = e^{-x}$. When y_1 is plugged in to the left side of the homogeneous problem we obtain

$$e^{-x} - 5 \cdot (-1)e^{-x} + 6e^{-x} = (1 + 5 + 6)e^{-x} = 12e^{-x}.$$

$$\boxed{\text{Thus } y_1 \text{ is NOT a solution of the homogeneous problem.}}$$

(b) Let $y = \frac{1}{12}e^{-x}$. We see that $y' = -\frac{1}{12}e^{-x}$ and $y'' = \frac{1}{12}e^{-x}$. When y is plugged in to the left side of the original problem we obtain

$$\frac{1}{12}e^{-x} - 5 \cdot (-1)\frac{1}{12}e^{-x} + 6\frac{1}{12}e^{-x} = \frac{1}{12}(1 + 5 + 6)e^{-x} = \frac{1}{12}12e^{-x} = e^{-x}.$$

$$\boxed{\text{Thus } y = \frac{1}{12}e^{-x} \text{ is a solution of the original problem.}}$$

(c) The general solution of the original problem is

$$\boxed{y = c_1e^{2x} + c_2e^{3x} + \tfrac{1}{12}e^{-x}.}$$