Math 242, Exam 1, Fall 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (7 points) **Find all constants r for which $y = e^{rx}$ is a solution of $3y'' + 3y' - 4y = 0$.**

Compute $y' = re^{rx}$ and $y'' = r^2e^{rx}$. So $y = e^{rx}$ is a solution of the DE provided

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0.$$

That is,

$$e^{rx}(3r^2 + 3r - 4) = 0.$$

The function e^{rx} is always positive and is never 0. So $y = e^{rx}$ is a solution of the differential equation if and only if $3r^2 + 3r - 4 = 0$. Use the quadratic formula to see that $\boxed{r = \dfrac{-3 \pm \sqrt{9+48}}{6}}$.

2. (7 points) **On the planet Gzyx, a ball dropped from a height of 40 ft hits the ground in 3 seconds. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit? I expect you to solve initial value problems. Unexplained, random formulas will not be accepted! (Recall that Newton's Law of Motion states that if $F(t)$ is the force acting on an object moving in a straight line at time t, m is the mass of the object, and $a(t)$ is the acceleration of the object at time t, then $F = ma$. The only force acting on this ball on planet Gzyx is the force of gravity and this force is constant.)**

Let $x(t)$ be the height of the ball above the ground at time t. Measure t in seconds and x in feet. We are told that $x''(t) = -k$ for some positive constant k. For the first event, we have $x(0) = 40$, $x'(0) = 0$, and $x(3) = 0$. For the second event, we have $x(0) = 200$ and $x'(0) = 0$. We want to find t_1 with $x(t_1) = 0$. We also want to find $x'(t_1)$.

We first think about the first event. Integrate to learn $x'(t) = -kt + C_1$. Plug in $x'(0) = 0$ to learn that $C_1 = 0$. Integrate again to learn $x(t) = -kt^2/2 + C_2$. Plug in $x(0) = 40$ to learn $C_2 = 40$. So, $x(t) = -kt^2/2 + 40$. Plug in $x(3) = 0$ to learn $k = \frac{80}{9}$.

Now turn to the second event. Integrate twice and evaluate the constants to learn that $x'(t) = -kt$ and $x(t) = -kt^2/2 + 200$; with $k = \frac{80}{9}$; so, $x(t) = -\frac{40}{9}t^2 + 200$. Solve $0 = x(t_1) = -\frac{40}{9}t_1^2 + 200$ to learn that $t_1 = \sqrt{45}$ and $x'(t_1) = -\frac{80}{9}\sqrt{45}$.

It takes the second ball $\sqrt{45}$ seconds to hit the ground. The ball is traveling downward at the speed $\frac{80}{9}\sqrt{45}$ feet per second when it hits the ground.

3. (7 points) **When the brakes are applied to a certain car, the acceleration of the car is $-k\frac{\text{m}}{\text{s}^2}$ for some positive constant k. Suppose that the car is traveling at the velocity $v_0\frac{\text{m}}{\text{s}}$ when the brakes are first applied and that the brakes continue to be applied until the car stops. I expect you to solve initial value problems. Unexplained, random formulas will not be accepted!**

(a) **Find the distance that the car travels between the moment that the brakes are first applied and the moment when the car stops. (Your answer will be expressed in terms of k and v_0.)**

Let $x(t)$ be the position of the car at time t. We take $t = 0$ to be the moment that the brakes are applied. So $v(0) = v_0$ and $x(0) = 0$. We are told $x'' = -k$. We integrate and plug in the points to see $v(t) = -kt + v_0$ and $x(t) = -kt^2/2 + v_0 t$. Let t_s be the time when the car stops. We have $0 = v(t_s) = -kt_s + v_0$. Thus, $t_s = v_0/k$. The distance traveled while the brakes were applied is

$$x(t_s) = x(v_0/k) = -k(v_0/k)^2/2 + v_0(v_0/k) = (v_0^2/k)(1 - 1/2) = \boxed{\frac{v_0^2}{2k}\text{m}}.$$

(b) **How does the stopping distance change if the initial velocity is changed to $5v_0$?**

$\boxed{\text{The stopping distance is multiplied by } 5^2}$, if v_0 is replaced by $5v_0$.

4. (8 points)

(a) **State the Existence and Uniqueness Theorem for first order differential equations.**

Consider the Initial Value Problem IVP: $y' = f(x, y)$ with $y(x_0) = y_0$.

(a) If f is continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a solution on some interval containing x_0 .

(b) If f and f_y are both continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a unique solution on some interval containing x_0 .

(b) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$\frac{dy}{dx} = 4x^3y - y \quad y(1) = -3?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = 4x^3y - y$. We see that f and $f_y = 4x^3 - 1$ are both continuous everywhere. We conclude that the given initial problem has a unique solution on some interval containing $x = 1$.

(c) **Solve the Initial Value Problem of part (b).** Separate the variables: $\int \frac{dy}{y} = \int (4x^3 - 1)dx$. Integrate to get $\ln|y| = x^4 - x + C$. Exponentiate to obtain $y = Ke^{x^4 - x}$, where $K = \pm e^C$. Plug in $y(1) = -3$ to see that $K = -3$. So the solution is

$$\boxed{y = -3e^{x^4 - x}}.$$

Check: We see that $y(1) = -3e^0 = -3$. ✓ We also see that

$$y' = -3e^{x^4 - x}(4x^3 - 1) = y(4x^3 - 1). \checkmark$$

(d) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$\frac{dy}{dx} = 4x^3y - y \quad y(1) = 0?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = 4x^3y - y$. We see that f and $f_y = 4x^3 - 1$ are both continuous everywhere. We conclude that the given initial problem has a unique solution on some interval containing $x = 1$.

(e) **Solve the Initial Value Problem of part (d).**

The unique solution of this Initial Value Problem is $y(x) = 0$ for all x . Notice that this function satisfies $\frac{dy}{dx} = 4x^3y - y$ because $0 = 0$ and this function satisfies $y(1) = 0$ because $0 = 0$. Notice also that the very first step in the solution of (c) is DIVIDE BOTH SIDES BY y and this is a good thing to do most of the time. However, it does not make sense when $y = 0$. The unique solution to this IVP is $y = 0$. So, the standard technique does not give the answer. It is a good thing that we know that the IVP has a unique solution. We step back, scratch our heads, and wonder what the solution is. By then the solution has jumped out at us.

5. (7 points) **A pitcher of buttermilk initially at $35°$ C is to be cooled by setting it on the front porch, where the temperature is $5°$ C. Suppose that the temperature of the buttermilk has dropped to $25°$ C after 20 minutes. When will the temperature of the buttermilk reach $10°$ C? I expect you to solve initial value problems. Unexplained, random formulas will not be accepted! (Recall that Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.)**

Let $T(t)$ be the temperature of the buttermilk at time t. We take $t = 0$ to be the moment that the buttermilk was put on the porch. We measure time in minutes and temperature in degrees C. We are told that $\frac{dT}{dt} = -k(T-5)$ for some positive constant k. We are also told that $T(0) = 35$ and $T(20) = 25$. We want t_{end} with $T(t_{\text{end}}) = 10$. We separate the variables and integrate to learn $\ln|T-5| = -kt+C$. Exponentiate to obtain $T-5 = Ke^{-kt}$, for $K = \pm e^C$. Plug in $T(0) = 35$ to learn that $K = 30$. Thus, $T-5 = 30e^{-kt}$ for all time t. We plug in $T(20) = 25$ to see that $25-5 = 30e^{-20k}$; hence $\frac{2}{3} = e^{-20k}$ and $\ln\frac{2}{3} = -20k$. So $\frac{\ln\frac{2}{3}}{-20} = k$ Now we can find t_{end} because $10 = T(t_{\text{end}}) = 5 + 30e^{-kt_{\text{end}}}$. Thus, $5 = 30e^{-kt_{\text{end}}}$ and $\frac{1}{6} = e^{-kt_{\text{end}}}$. Take the logarithm of both sides to obtain

$$\ln(\tfrac{1}{6}) = -kt_{\text{end}}.$$

Thus,

$$t_{\text{end}} = -\ln(\tfrac{1}{6})\tfrac{1}{k} = -\ln\tfrac{1}{6}\tfrac{-20}{\ln(\frac{2}{3})} = \ln 6\tfrac{20}{\ln(\frac{3}{2})} = \boxed{\frac{20\ln(6)}{\ln 3 - \ln 2}\text{min}}.$$

6. (7 points) **Find the general solution of $(x^2+1)\frac{dy}{dx} + 3xy = 6x$. Express your answer in the form $y = y(x)$. Check your answer.**

This DE is a First Order Linear problem. Divide both sides by x^2+1 to obtain:

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}.$$

Multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{\frac{3}{2}}$$

to obtain

$$(x^2+1)^{\frac{3}{2}}\frac{dy}{dx} + 3x\sqrt{x^2+1}y = 6x\sqrt{x^2+1}.$$

The above problem is

$$\tfrac{d}{dx}((x^2+1)^{\frac{3}{2}}y) = 6x\sqrt{x^2+1}.$$

Integrate both sides with respect to x to obtain

$$(x^2+1)^{\frac{3}{2}}y = 2(x^2+1)^{\frac{3}{2}} + C.$$

Multiply both sides by $(x^2+1)^{-\frac{3}{2}}$ to get

$$\boxed{y = 2 + C(x^2+1)^{-\frac{3}{2}}}.$$

Check: We compute

$$(x^2+1)\frac{dy}{dx} + 3xy$$

$$= (x^2+1)\left(-\frac{3}{2}C(x^2+1)^{-\frac{5}{2}}2x\right) + 3x\left(2 + C(x^2+1)^{-\frac{3}{2}}\right)$$

$$= -3C(x^2+1)^{-\frac{3}{2}}x + 3x\left(2 + C(x^2+1)^{-\frac{3}{2}}\right) = 6x. \checkmark$$

7. (7 points) **A 120-gallon (gal) tank initially contains 90 pounds (lb) of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when the tank becomes full?**

Let $x(t)$ be the number of pounds of salt in the tank at time t. We are told that $x(0) = 90\text{lbs}$. We know that salt is entering the tank at the rate of $2\frac{\text{lbs}}{\text{gal}} \times \frac{4\text{gal}}{\text{min}} = 8\frac{\text{lbs}}{\text{min}}$ and salt is leaving the tank at the rate of $\frac{x}{90+t}\frac{\text{lbs}}{\text{gal}}\frac{3\text{gal}}{\text{min}} = \frac{3x}{90+t}\frac{\text{lbs}}{\text{min}}$. We must solve the initial value problem:

$$(*) \qquad \frac{dx}{dt} = 8 - \frac{3x}{90+t}, \quad x(0) = 90.$$

This is a First Order Linear Differential Equation:

$$\frac{dx}{dt} + \frac{3x}{90+t} = 8.$$

Multiply both sides by

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3$$

to obtain

$$(90+t)^3\frac{dx}{dt} + 3(90+t)^2x = 8(90+t)^3.$$

This is

$$\frac{d}{dt}\left((90+t)^3 x\right) = 8(90+t)^3.$$

Integrate both sides with respect to t to obtain

$$(90+t)^3 x = 2(90+t)^4 + C.$$

Multiply both sides by $(90+t)^{-3}$ to get

$$x(t) = 2(90+t) + C(90+t)^{-3}.$$

Plug in $x(0) = 90$ to learn

$$90 = 2(90) + C(90)^{-3}$$

$$-(90) = C(90)^{-3}$$

$$-(90)^4 = C.$$

So, $x(t) = 2(90+t) - (90)^4(90+t)^{-3}$. The tank becomes full at $t = 30$. We see that $x(30) = \boxed{2(120) - (90)^4(120)^{-3}\text{pounds of salt}}$.

Check: Our solution $x(t) = 2(90+t) - (90)^4(90+t)^{-3}$ does satisfy the IVP (*). Indeed, $\frac{dx}{dt} = 2 - 3(90)^4(90+t)^{-4}$ and

$$8 - \frac{3x}{90+t} = 8 - \frac{3(2(90+t) - (90)^4(90+t)^{-3})}{90+t} = 8 - 6 + 3\frac{(90)^4}{(90+t)^4}.$$

These expressions are equal as desired. Also, $x(0) = 2(90) - (90)^4/(90)^3 = 90$.