Math 242, Exam 1, Spring 2010 SOLUTIONS

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems **ON TWO SIDES**. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Solve** $yy'+x=\sqrt{x^2+y^2}$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This is a homogeneous problem. Divide both sides by x to write the problem as

$$\frac{y}{x}y'+1=\sqrt{1+\left(\frac{y}{x}\right)^2}.$$

Let $v=\frac{y}{x}$. In other words, $xv=y$. Take the derivative with respect to x to see that $xv'+v=y'$. We must solve

$$v(xv'+v)+1=\sqrt{1+v^2}.$$

We must solve

$$xv\frac{dv}{dx}=\sqrt{1+v^2}-v^2-1.$$

We must solve

$$v\frac{dv}{\sqrt{1+v^2}-v^2-1}=\frac{dx}{x}.$$

Integrate both sides. Let $w=1+v^2$. It follows that $dw=2vdv$. We must solve

$$\frac{1}{2}\int\frac{dw}{\sqrt{w}-w}=\ln|x|+C.$$

We have

$$\ln|x|+C=\frac{1}{2}\int\frac{dw}{\sqrt{w}(1-\sqrt{w})}.$$

Let $u=\sqrt{w}$. We have $du=\frac{1}{2}w^{-1/2}dw$.

We have

$$\ln|x| + C = \int \frac{du}{1-u} = -\ln|1-u| = -\ln|1-\sqrt{w}| = -\ln|1-\sqrt{1+v^2}|$$

$$= -\ln\left|1-\sqrt{1+\left(\frac{y}{x}\right)^2}\right| = -\ln\left|\frac{x-\sqrt{x^2+y^2}}{x}\right| = -\ln\left|x-\sqrt{x^2+y^2}\right| + \ln|x|.$$

Subtract $\ln|x|$ from both sides:

$$C = -\ln\left|x-\sqrt{x^2+y^2}\right|$$

or

$$\ln\left|x-\sqrt{x^2+y^2}\right| = -C$$

Exponentiate. Let K be the new constant e^{-C}. We have

$$x-\sqrt{x^2+y^2} = K;$$

so $x-K=\sqrt{x^2+y^2}$ and $(x-K)^2 = x^2+y^2$ and $\boxed{\pm\sqrt{(x-K)^2-x^2} = y.}$

Check: We check $y = +\sqrt{(x-K)^2-x^2}$, with $K \le x$. We see that

$$y' = \frac{2(x-K)-2x}{2\sqrt{(x-K)^2-x^2}} = \frac{-K}{\sqrt{(x-K)^2-x^2}}.$$

So, $yy'+x = -K+x$. On the other hand,

$$\sqrt{x^2+y^2} = \sqrt{x^2+(x-K)^2-x^2} = \sqrt{(x-K)^2} = x-K.$$

Thus, $yy'+x = \sqrt{y^2+x^2}$ as required. ✓

2. **Solve** $y' = y+y^3$**. Express your answer in the form** $y(x)$**. Check your answer.**

This is a Bernoulli equation. Let $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ or $\frac{y^3v'}{-2} = y'$. At this moment our equation looks like $\frac{y^3v'}{-2} = y+y^3$. Multiply by $-2y^{-3}$ to get $v' = -2y^{-2}-2$, which is $v'+2v = -2$. This is a first order linear DE. The integrating factor is $\mu = e^{\int 2dx} = e^{2x}$. Multiply both sides by e^{2x}. We must solve: $e^{2x}v' + 2e^{2x}v = -2e^{2x}$, which is $\frac{d(e^{2x}v)}{dx} = -2e^{2x}$. Integrate both sides with respect to x to get: $e^{2x}v = -e^{2x} + C$. So $v = -1 + Ce^{-2x}$; that is, $y^{-2} = -1 + Ce^{-2x}$ or $\boxed{y = (-1 + Ce^{-2x})^{-1/2}}$.

Check: We see that

$$y' = (-1/2)(-1 + Ce^{-2x})^{-3/2}(-2Ce^{-2x}) = (-1 + Ce^{-2x})^{-3/2}Ce^{-2x}.$$

On the other hand,

$$y + y^3 = (-1+Ce^{-2x})^{-1/2} + (-1+Ce^{-2x})^{-3/2} = (-1+Ce^{-2x})^{-3/2}(-1+Ce^{-2x}+1)$$

$$= (-1 + Ce^{-2x})^{-3/2}Ce^{-2x}.$$

These agree. ✓

3. **A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture — kept uniform by stirring — is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?**

Let $x(t)$ be the number of kg of salt in the tank at time t seconds. The problem tells us that $\frac{dx}{dt} = 0 - \frac{5L}{s}\frac{x\text{kg}}{1000L}$. So, $\frac{dx}{dt} = \frac{x}{-200}$. Separate the variables: $\frac{dx}{x} = \frac{-1}{200}dt$. Integrate to see $\ln x = \frac{-1}{200}t + C$ or $x = Ke^{\frac{-1}{200}t}$, where K is the constant e^C. The problem tells us that $100 = x(0) = K$; so, $x(t) = 100e^{\frac{-1}{200}t}$. We are supposed to find the time when $10 = x(t)$. So we solve for t: $10 = 100e^{\frac{-1}{200}t}$ which becomes $\frac{1}{10} = e^{\frac{-1}{200}t}$. Take ln of both sides to get $-\ln(10) = \frac{-1}{200}t$, or $\boxed{t = 200 \ln 10 \text{ seconds}}$.

4.

(a) **State the Existence and Uniqueness Theorem for first order differential equations.**

Consider the Initial Value Problem IVP: $y' = f(x, y)$ with $y(x_0) = y_0$.

(a) If f is continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a solution on some interval containing x_0.

(b) If f and f_y are both continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a unique solution on some interval containing x_0.

(b) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$(1+x^2)y' = (2+y)^2 \quad y(0) = 0?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = \frac{(2+y)^2}{(1+x^2)}$. We see that f and $f_y = \frac{2(2+y)}{(1+x^2)}$ are both continuous everywhere. We conclude the given initial problem has a unique solution on some interval containing $x = 0$.

(c) **Solve the Initial Value Problem of part (b).** Separate the variables: $\int \frac{dy}{(2+y)^2} = \int \frac{dx}{x^2+1}$. Integrate to get $-(2+y)^{-1} = \arctan x + C$. Plug in $y(0) = 0$ to see that $C = -\frac{1}{2}$. So the solution is

$$\frac{-1}{\arctan x - \frac{1}{2}} - 2 = y$$

or

$$\boxed{\frac{2}{1 - 2\arctan x} - 2 = y}.$$

Check: We see that $y(0) = 2 - 2 = 0$. ✓ We also see that

$$y' = \frac{-2}{(1-2\arctan x)^2}(-2)\frac{1}{1+x^2};$$

so,

$$(1+x^2)y' = \frac{4}{(1-2\arctan x)^2}.$$

On the other hand,

$$(y+2)^2 = \left(\frac{2}{1-2\arctan x}\right)^2.$$

These agree. ✓

(d) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$(1+x^2)y' = (2+y)^2 \quad y(0) = -2?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = \frac{(2+y)^2}{(1+x^2)}$. We see that f and $f_y = \frac{2(2+y)}{(1+x^2)}$ are both continuous everywhere. We conclude the given initial problem has a unique solution on some interval containing $x = 0$.

(e) **Solve the Initial Value Problem of part (d).** The technique we used in (c) does not work because we can not divide by $y+2$ if y is sometimes equal to -2. On the other hand, we are guaranteed that a unique solution exists on some interval near $x = 0$. We have $(0,-2)$ on our solution and we we leave this point we leave with slope 0 because $y' = \frac{(2+y)^2}{(1+x^2)}$ so $y'(0,-2) = 0$. Travel a little bit along the line $y = -2$. Ask the DE which way you should go. That is, plug (near $0,-2$) into y'. Again, $y' = 0$. In fact the unique solution to this IVP is $\boxed{y = -2}$. This function satisfies the initial condition and y' is always zero so $(1+x^2)0 = (0)^2$ does indeed hold!

5. **When the brakes are applied to a certain car, the acceleration of the car is $-k$ m/s^2 for some positive constant k. Suppose that the car is traveling at the velocity v_0 m/s when the brakes are first applied and that the brakes continue to be applied until the car stops.**

(a) **Find the distance that the car travels between the moment that the brakes are first applied and the moment when the car stops. (Your answer will be expressed in terms of k and v_0.)**

Let $x(t)$ be the position of the car at time t. We take $t = 0$ to be the moment that the brakes are applied. So $v(0) = v_0$ and $x(0) = 0$. We are told $x'' = -k$. We integrate and plug in the points to see $v(t) = -kt + v_0$ and $x(t) = -kt^2/2 + v_0 t$. Let t_s be the time when the car stops. We have $0 = v(t_s) = -kt_s + v_0$. Thus, $t_s = v_0/k$. The distance traveled while the brakes were applied is

$$x(t_s) = x(v_0/k) = -k(v_0/k)^2/2 + v_0(v_0/k) = (v_0^2/k)(1 - 1/2) = \boxed{\frac{v_0^2}{2k}}.$$

(b) **How does the stopping distance change if the initial velocity is changed to $3v_0$?**

$\boxed{\text{The stopping distance is multiplied by } 3^2}$ is v_0 is replaced by $3v_0$.