Math 242, Exam 1, Spring, 2018

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators or Cell phones.

(1) **Solve the initial value problem $\frac{dy}{dx} = ye^x$, $y(0) = 2e$. Express your answer in the form $y = y(x)$. Please check your answer.**

Separate the variables and integrate:

$$\int \frac{dy}{y} = \int e^x dx$$

$$\int \ln|y| = e^x + C$$

Exponentiate both sides:

$$|y| = e^{e^x + C}$$

$$y = \pm e^C e^{e^x}$$

Let K denote the constant $\pm e^C$.

$$y = Ke^{e^x}.$$

Plug in the initial condition:

$$2e = y(0) = Ke$$

So, $K = 2$ and the answer is $\boxed{y = 2e^{e^x}.}$

<u>Check</u> $y(0) = 2e^{e^0} = 2e^1 = 2e\checkmark$.

$$\frac{dy}{dx} = 2e^{e^x}e^x = ye^x.\checkmark$$

(2) **Solve $3y^2\frac{dy}{dx} + y^3 = e^{-x}$. Express your answer in the form $y = y(x)$. Please check your answer.**

Write the equation as

$$3\frac{dy}{dx} + y = e^{-x}y^{-2}.$$

This is a Bernoulli equation. Let $v = y^{1-(-2)} = y^3$. Observe that $\frac{dv}{dx} = 3y^2\frac{dy}{dx}$. Multiply both sides by y^2:

$$3y^2\frac{dy}{dx} + y^3 = e^{-x}$$

$$\frac{dv}{dx} + v = e^{-x}$$

This is a first order linear DE. Multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int 1dx} = e^x.$$

The DE is

$$\frac{dv}{dx}e^x + ve^x = 1.$$

The left side is $\frac{d}{dx}(e^x v)$. Integrate both sides:

$$e^x v = x + C.$$

$$v = xe^{-x} + Ce^{-x}$$

$$y^3 = xe^{-x} + Ce^{-x}$$

$$\boxed{y = (xe^{-x} + Ce^{-x})^{1/3}.}$$

Check Plug y into the left side of the DE to obtain

$$3(xe^{-x}+Ce^{-x})^{2/3}(1/3)(xe^{-x}+Ce^{-x})^{-2/3}(-xe^{-x}+e^{-x}-Ce^{-x})+(xe^{-x}+Ce^{-x})$$

$$= (-xe^{-x} + e^{-x} - Ce^{-x}) + (xe^{-x} + Ce^{-x}) = e^{-x}\checkmark$$

(3) **Consider the initial value problem $\frac{dy}{dx} = x + \frac{2}{y}$, $y(1) = 3$. Use Euler's method to approximate $y(12/10)$. Use two steps, each of size $1/10$.**

Let $F(x, y) = x + \frac{2}{y}$, $x_0 = 1$, $x_1 = 11/10$, $x_2 = 12/10$, and $y_0 = 3$. We find y_1 and y_2 so that

$$\frac{y_1 - y_0}{x_1 - x_0} = F(x_0, y_0)$$

and

$$\frac{y_2 - y_1}{x_2 - x_1} = F(x_1, y_1)$$

Then y_2 is our approximation of $y(12/10)$. We have

$$y_1 = 3 + (1/10)(1 + 2/3) = 3 + 1/6 = 19/6$$

and

$$y_2 = y_1 + (1/10)(x_1 + \frac{2}{y_1}) = 19/6 + (1/10)(11/10 + 12/19).$$

$$\boxed{\text{Our approximation of } y(12/10) \text{ is } y_2 = 19/6 + (1/10)(11/10 + 12/19).}$$

(4) **A 120-gallon tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal flows into the tank at the rate of 4 gal/min and the well-mixed mixture flows out of the tank at the rate of 3 gal/min. How much salt is in the tank when the tank is full?**

Let $x(t)$ be the number of pounds of salt in the tank at time t. We are told that $x(0) = 90$. We are also told that

$$\frac{dx}{dt} = 2\frac{\text{lb}}{\text{gal}}4\frac{\text{gal}}{\text{min}} - \frac{x}{90+t}\frac{\text{lb}}{\text{gal}}3\frac{\text{gal}}{\text{min}}.$$

We are supposed to find $x(30)$.

The DE

$$\frac{dx}{dt} + \frac{3x}{90+t} = 8$$

is a first order linear DE. Multiply both sides by

$$\mu(t) = e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3$$

to obtain

$$(90+t)^3\frac{dx}{dt} + 3x(90+t)^2 = 8(90+t)^3$$

Observe that the left side is $\frac{d}{dt}((90+t)^3x)$. Integrate both sides to obtain

$$(90+t)^3x = 2(90+t)^4 + C.$$

Thus,

$$x = 2(90+t) + C(90+t)^{-3}.$$

Plug in $t = 0$ to see

$$90 = x(0) = 180 + C(90)^{-3}$$
$$-(90)^4 = C$$

There are $x(30) = 2(90+30) - (90)^4(90+30)^{-3}$ pounds of salt in the tank when the tank is full.

(5) **The acceleration of a car is proportional to the difference between 200 ft/sec and the velocity of the car. If this car can accelerate from 0 to 40 ft/sec in 2 seconds, how long will it take for the car to accelerate from rest to 150 ft/sec? Set up an initial value problem. Solve the initial value problem.**

Let $v(t)$ be the velocity of the car at time t. We are told $v(0) = 0$, $v(2) = 40$, and $\frac{dv}{dt} = k(200-v)$. We are supposed to find t with $v(t) = 150$. Separate the variables

$$\int \frac{dv}{200-v} = \int kdt$$
$$-\ln(200-v) = kt + C$$
$$\frac{1}{200-v} = e^C e^{kt}$$

$$\frac{1}{e^C e^{kt}} = 200 - v$$

$$v = 200 - e^{-C} e^{-kt}$$

Plug in $t = 0$:

$$0 = v(0) = 200 - e^{-C}$$

so $e^{-C} = 200$

$$v = 200(1 - e^{-kt})$$

Plug in $v(2)$.

$$40 = v(2) = 200(1 - e^{-2k})$$

$$1/5 = 1 - e^{-2k}$$

$$e^{-2k} = 1 - 1/5$$

$$-2k = \ln(4/5)$$

$$k = (-1/2)\ln(4/5)$$

$v(t) = 150$ when $150 = 200(1 - e^{-kt})$; so $3/4 = 1 - e^{-kt}$

$$e^{-kt} = 1/4$$

$$-kt = \ln(1/4)$$

$$t = \frac{\ln(1/4)}{-k} = \frac{\ln(1/4)}{-(-1/2)\ln(4/5)}$$

The car reaches 150 ft/sec at time $\boxed{\frac{2\ln 4}{\ln(5/4)}}$ seconds.