Math 242, Exam 1, Solutions, Fall, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **A population of 80 cougars decreases at a rate of 5% per year. How many cougars will there be after 6 years?**

Let $P(t)$ be the number of cougars at time t. We are told that $\frac{dP}{dt} = -.05P$ and $P(0) = 80$.

Solve the inital value problem; get $P(t) = 80e^{-.05t}$. The number of cougars after 6 years is $\boxed{P(6) = 80e^{-6(.05)}}$.

(2) **Set this problem up completely. DO NOT SOLVE IT. A 120-gallon tank initially contains 90 pounds of salt dissolved in 90 gallons of water. Brine containing 2 pounds/gallon of salt flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 3 gallons per minute. How much salt does the tank contain when it is full?**

Answer: Let $x(t)$ be the number of pounds of salt in the tank at time t. We are told that $x(0) = 90$. We are also told that

$$\frac{dx}{dt} = 2\frac{\text{lbs}}{\text{gal}} \times 4\frac{\text{gal}}{\text{min}} - \frac{x}{90+t}\frac{\text{lbs}}{\text{gal}} \times 3\frac{\text{gal}}{\text{min}}.$$

We are supposed to solve the Initial Value Problem

$$\begin{cases} \frac{dx}{dt} = 8 - 3\frac{x}{90+t} \\ x(0) = 90 \end{cases}$$

This will give us a function $x(t) = ...$ The answer is $x(30)$.

Explanation: Notice that $90 + t$ is the amount of brine in the tank at time t. (There are 90 gallons in the tank at $t = 0$; 91 gallons at $t = 1$, 92 gallons at $t = 2$. The tank is full at $t = 30$.)

(3) At time zero an obect has position x_0 and velocity v_0. Suppose that the object moves through a resisting medium with resistance proportional to its velocity v, so that $\frac{dv}{dt} = -kv$. Find the velocity and position of the object at time t.

Let $x(t)$ be the position of the object at time t. It follows that $v(t) = \frac{dx}{dt}$ is the velocity of the object at time t.

We first solve the Initial Value Problem

$$\frac{dv}{dt} = -kv, \quad v(0) = v_0.$$

Separate the variables and integrate:

$$\int \frac{dv}{v} = -\int kdt.$$

$$\ln|v| = -kt + C$$

$$|v| = e^C e^{-kt}$$

$$v = \pm e^C d^{-kt}$$

Plug in $t = 0$ to see that

$$v_0 = v(0) = \pm e^C,$$

$$\boxed{v(t) = v_0 e^{-kt}}$$ This is part of our answer.

Now we solve the Inital Value Problem

$$\frac{dx}{dt} = v_0 e^{-kt}, \quad x(0) = x_0.$$

We separate the variables and integrate

$$\int dx = \int v_0 e^{-kt} dt$$

$$x(t) = \frac{v_0}{-k} e^{-kt} + C_2$$

Plug in $t = 0$ to see that

$$x_0 = x(0) = \frac{v_0}{-k} + C_2$$

So

$$x_0 + \frac{v_0}{k} = C_2$$

and

$$x(t) = \frac{v_0}{-k} e^{-kt} + x_0 + \frac{v_0}{k}.$$

In other words,

$$\boxed{x(t) = \frac{v_0}{k}(1 - e^{-kt}) + x_0}.$$ This is the rest of our answer.

(4) **Solve the Differential Equation $xy' + 2y = 6x^2\sqrt{y}$.**

This is a Bernoulli equation with $n = 1/2$. We let $v = y^{1-n}$; so,

$$v = y^{1/2}.$$

It follows that $\frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$. Multiply both sides of the original equation by $\frac{1}{2}y^{-1/2}$ to obtain

$$\begin{aligned} x(\tfrac{1}{2})y^{-1/2}y' + y^{1/2} &= 3x^2 \\ x\tfrac{dv}{dx} + v &= 3x^2 \\ \tfrac{dv}{dx} + \tfrac{1}{x}v &= 3x. \end{aligned} \tag{1}$$

This is a First Order Linear Differential Equation. We multiply both sides of (1) by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

to obtain

$$x\tfrac{dv}{dx} + v = 3x^2$$

The left side of the most recent equation is $\frac{d}{dx}(xv)$. To solve

$$\tfrac{d}{dx}(xv) = 3x^2$$

we integrate both sides with respect to x. Thus,

$$xv = x^3 + C.$$

Thus,

$$v = x^2 + \frac{C}{x}$$

$$y^{1/2} = x^2 + \frac{C}{x}$$

$$\boxed{y = \left(x^2 + \frac{C}{x}\right)^2.}$$

(5) **The Logistic Equation is $\frac{dP}{dt} = kP(M-P)$, where k and M are positive constants. The solution of the Logistic Equation is**

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-kMt}}.$$

Recall that if a population $P(t)$ satisfies the logistic equation

$$\frac{dP}{dt} = aP - bP^2,$$

where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur, then the limiting population is

$$M = \lim_{t\to\infty} P(t) = \frac{B(0)P(0)}{D(0)}.$$

Consider a rabbit population $P(t)$ which satisfies the logistic equation. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

We are told that $P(0) = 240$, $B(0) = 9$, $D(0) = 12$. We calculate

$$M = \frac{B(0)P(0)}{D(0)} = \frac{9(240)}{12} = 180$$

and

$$k = b = \frac{D(0)}{P(0)^2} = \frac{12}{(240)^2} = \frac{1}{(240)(20)}.$$

We must find t so that

$$\frac{105}{100}(180) = \frac{180(240)}{240 + (180 - 240)e^{-180t/((240)20)}}.$$

Cancel 180 from the left and the right. Divide top and bottom on the right by 60.

$$\frac{105}{100} = \frac{4}{4 - e^{-(3/80)t}}$$

$$4 - e^{-(3/80)t} = 4\left(\frac{100}{105}\right)$$

$$4 - 4\left(\frac{100}{105}\right) = e^{-(3/80)t}$$

$$4\left(\frac{5}{105}\right) = e^{-(3/80)t}$$

$$\ln\left(\frac{20}{105}\right) = -(3/80)t$$

$$\ln\left(\frac{105}{20}\right) = (3/80)t$$

$$\boxed{\frac{80}{3}\ln\left(\frac{105}{20}\right) \text{ months} = t}.$$