Math 242, Exam 3, Spring, 2024 Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the general solution of $x^3 + 3y - xy' = 0$. (In this problem $y = y(x)$.)**

This is a First Order Linear problem. Write it in the form

$$-xy' + 3y = -x^3.$$

Divide each side by $-x$:

$$y' + \tfrac{-3}{x}y = x^2.$$

Multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-3}{x}} = e^{-3\ln x} = x^{-3} :$$

$$x^{-3}y' - 3x^{-4}y = \tfrac{1}{x}.$$

Observe that the left side is equal to $\frac{d}{dx}(x^{-3}y)$. Integrate both sides

$$x^{-3}y = \ln|x| + C$$

$$\boxed{y = x^3(\ln|x| + C).}$$

Check. Plug

$$y = x^3(\ln|x| + C)$$
$$y' = x^3(\tfrac{1}{x}) + 3x^2(\ln|x| + C)$$

into $x^3 + 3y - xy'$ and obtain

$$x^3 + 3x^3(\ln|x| + C) - x\Big(x^2 + 3x^2(\ln|x| + C)\Big)$$

$$= x^3 + 3x^3(\ln|x| + C) - \Big(x^3 + 3x^3(\ln|x| + C)\Big) = 0.\checkmark$$

(2) **Find the general solution of $9y'' - 6y' + y = 0$. (In this problem $y = y(x)$.)**

We try $y = e^{rx}$. We consider the characteristic equation

$$9r^2 - 6r + 1 = 0$$

$$(3r - 1)^2 = 0$$

The general solution of the Differential Equation is

$$\boxed{y = (c_1 + xc_2)e^{r/3}.}$$

Check. We plug

$$\begin{aligned} y &= (c_1 + xc_2)e^{r/3} \\ y' &= \tfrac{1}{3}(c_1 + xc_2)e^{r/3} + c_2 e^{r/3} \\ &= \tfrac{1}{3}(c_1 + 3c_2 + xc_2)e^{r/3} \\ y'' &= \tfrac{1}{9}(c_1 + 3c_2 + xc_2)e^{r/3} + \tfrac{1}{3}c_2 e^{r/3} \\ &= \tfrac{1}{9}(c_1 + 6c_2 + xc_2)e^{r/3} \end{aligned}$$

into $9y'' - 6y' + y$ and obtain

$$\left\{ \begin{array}{l} (c_1 + 6c_2 + xc_2)e^{r/3} \\ -2(c_1 + 3c_2 + xc_2)e^{r/3} \\ +(c_1 + xc_2)e^{r/3} \end{array} \right\} = 0. \checkmark$$

(3) **Find the general solution of $y'' - 4y' + 5y = 0$. (In this problem $y = y(x)$.)**

We try $y = e^{rx}$. We consider the characteristic equation $r^2 - 4r + 5 = 0$. We use the quadratic formula. The roots of $ar^2 + br + c = 0$ are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Thus, the roots of $r^2 - 4r + 5 = 0$ are

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

The general solution of the Differential Equation is

$$\boxed{y = c_1 e^{2x} \sin x + c_2 e^{2x} \cos x.}$$

Check. Plug

$$\begin{aligned} y &= c_1e^{2x}\sin x + c_2e^{2x}\cos x \\ y' &= c_1e^{2x}\cos x - c_2e^{2x}\sin x + 2c_1e^{2x}\sin x + 2c_2e^{2x}\cos x \\ &= (c_1+2c_2)e^{2x}\cos x + (-c_2+2c_1)e^{2x}\sin x \\ y'' &= \begin{cases} -(c_1+2c_2)e^{2x}\sin x + (-c_2+2c_1)e^{2x}\cos x \\ +2(c_1+2c_2)e^{2x}\cos x + 2(-c_2+2c_1)e^{2x}\sin x \end{cases} \\ &= +(4c_1+3c_2)e^{2x}\cos x + (3c_1-4c_2)e^{2x}\sin x \end{aligned}$$

into $y''-4y'+5y$ and obtain

$$\begin{cases} (4c_1+3c_2)e^{2x}\cos x + (3c_1-4c_2)e^{2x}\sin x \\ -4\Big((c_1+2c_2)e^{2x}\cos x + (-c_2+2c_1)e^{2x}\sin x\Big) \\ +5\Big(c_1e^{2x}\sin x + c_2e^{2x}\cos x\Big) \end{cases}$$

$$= \Big((4c_1+3c_2)-4(c_1+2c_2)+5c_2e^{2x}\Big)e^{2x}\cos x + \Big((3c_1-4c_2)-4(-c_2+2c_1)+5c_1\Big)e^{2x}\sin x$$
$$= 0 \checkmark$$

(4) **Find a particular solution of $y''+y'+y=\cos 2x$. (In this problem $y=y(x)$.)**

We try $y = A\sin 2x + B\cos 2x$. We plug

$$\begin{aligned} y &= A\sin 2x + B\cos 2x \\ y' &= 2A\cos 2x - 2B\sin 2x \\ y'' &= -4A\sin 2x - 4B\cos 2x \end{aligned}$$

into $y''+y'+y=\cos 2x$ and obtain

$$\left.\begin{cases} (-4A\sin 2x - 4B\cos 2x) \\ +(2A\cos 2x - 2B\sin 2x) \\ +(A\sin 2x + B\cos 2x) \end{cases}\right\} = \cos 2x.$$

$$(-4A-2B+A)\sin 2x + (-4B+2A+B)\cos 2x = \cos 2x.$$

We hope to find A and B with

$$\begin{cases} 0 = -3A - 2B \\ 1 = 2A - 3B \end{cases}$$

Replace Equation 2 with Equation 2 plus $2/3$ times Equation 1:

$$\begin{cases} 0 = -3A - 2B \\ 1 = \qquad -\frac{13}{3}B \end{cases}$$

Therefore, $B = \frac{-3}{13}$ and $A = \frac{2}{13}$. We conclude that

$$\boxed{y = \tfrac{1}{13}(2\sin 2x - 3\cos 2x)}$$

is a particular solution of $y'' + y' + y = \cos 2x$.

Check. Plug

$$\begin{aligned} y &= \tfrac{1}{13}(2\sin 2x - 3\cos 2x) \\ y' &= \tfrac{1}{13}(4\cos 2x + 6\sin 2x) \\ y'' & \tfrac{1}{13}(-8\sin 2x + 12\cos 2x) \end{aligned}$$

into $y'' + y' + y$ and obtain

$$\frac{1}{13}\begin{cases} -8\sin 2x + 12\cos 2x \\ +4\cos 2x + 6\sin 2x \\ +2\sin 2x - 3\cos 2x \end{cases}$$

$$= \frac{1}{13}\Big((-8+6+2)\sin 2x + (12+4-3)\cos 2x\Big) = \cos 2x. \checkmark$$

(5) **At time zero an object has position x_0 and velocity v_0. Suppose that the object moves through a resisting medium with resistance proportional to its velocity v, so that $\frac{dv}{dt} = -kv$. Find the velocity and position of the object at time t.**

Let $x(t)$ be the position of the object at time t. It follows that $v(t) = \frac{dx}{dt}$ is the velocity of the object at time t.

We first solve the Initial Value Problem

$$\frac{dv}{dt} = -kv, \quad v(0) = v_0.$$

Separate the variables and integrate:

$$\int \frac{dv}{v} = -\int k dt.$$

$$\begin{aligned} \ln|v| &= -kt + C \\ |v| &= e^C e^{-kt} \\ v &= \pm e^C d^{-kt} \end{aligned}$$

Plug in $t = 0$ to see that

$$v_0 = v(0) = \pm e^C,$$

$$\boxed{v(t) = v_0 e^{-kt}} \quad \text{This is part of our answer.}$$

Now we solve the Inital Value Problem

$$\frac{dx}{dt} = v_0 e^{-kt}, \quad x(0) = x_0.$$

We separate the variables and integrate

$$\int dx = \int v_0 e^{-kt} dt$$

$$x(t) = \frac{v_0}{-k} e^{-kt} + C_2$$

Plug in $t = 0$ to see that

$$x_0 = x(0) = \frac{v_0}{-k} + C_2$$

So

$$x_0 + \frac{v_0}{k} = C_2$$

and

$$x(t) = \frac{v_0}{-k} e^{-kt} + x_0 + \frac{v_0}{k}.$$

In other words,

$$\boxed{x(t) = \frac{v_0}{k}(1 - e^{-kt}) + x_0}. \quad \text{This is the rest of our answer.}$$