Math 242, Final Exam, Fall, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 100 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators, Cell phones, computers, notes, etc.

(1) **Newton's Law of Cooling states that the rate of change with respect to time of the temperature $T(t)$ of an object is proportional to the difference between T and the temperature A of the surrounding medium. A four pound roast, initially at $50°$F, is placed in a $375°$F oven at 5:00 P.M. After 75 minutes it is found that the temperature $T(t)$ of the roast is $125°$F. When will the roast be $150°$F?**

In this problem $T(t)$ is the temperature of the roast at time t, where T is measured in degrees F and t is measured in minutes with $t = 0$ representing 5:00 PM. The surrounding medium is the air in the oven whose temperature is 375. So we are given

$$\frac{dT}{dt} = k(T - 375)$$
$$T(0) = 50$$
$$T(75) = 125$$

We separate the variables and integrate to see that:

$$\int \frac{dT}{T-375} = \int k\,dt \quad \text{and}$$

$$\ln|T - 375| = kt + C.$$

Exponentiate to see that

$$|T - 375| = e^C e^{kt} \quad \text{or} \quad T - 375 = \pm e^C e^{kt}.$$

Let K be the name of the constant $\pm e^C$. We have learned that

$$T = 375 + Ke^{kt}.$$

Plug in $T(0) = 50$ to see that $K = -325$. Thus

$$T = 375 - 325e^{kt}.$$

Plug in $T(75) = 125$ to see that

$$125 = 375 - 325e^{75k},$$
$$325e^{75k} = 250,$$
$$e^{75k} = \frac{10}{13},$$
$$75k = \ln\left(\frac{10}{13}\right), \quad \text{and}$$
$$k = \frac{1}{75}\ln\left(\frac{10}{13}\right).$$

At this point we know that

$$T(t) = 375 - 325e^{t\frac{1}{75}\ln(\frac{10}{13})}.$$

Now we find the t with $T(t) = 150$. In other words, we solve

$$150 = 375 - 325e^{t\frac{1}{75}\ln(\frac{10}{13})}$$
$$325e^{t\frac{1}{75}\ln(\frac{10}{13})} = 225$$
$$e^{t\frac{1}{75}\ln(\frac{10}{13})} = \frac{225}{325}$$
$$t\frac{1}{75}\ln(\frac{10}{13}) = \ln(\frac{9}{13})$$
$$t = 75\frac{\ln(\frac{9}{13})}{\ln(\frac{10}{13})}$$

The roast will be $150°$ at $75\frac{\ln(\frac{9}{13})}{\ln(\frac{10}{13})}$ minutes after 5:00.

(2) **A 120-gallon tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well stirred mixture flows out of the tank at the rate of 3 gal/min. Set up an initial Value problem for the the number of pounds $x(t)$ of salt in the tank at time t for $0 \leq t \leq 30$, but DO NOT SOLVE the Initial Value Problem.**

The rate at which salt flows into the tank is

$$2\frac{\text{lb}}{\text{g}} \times 4\frac{\text{g}}{\text{min}} = 8\frac{\text{lb}}{\text{min}}.$$

The rate at which salt flows out of the tank is

$$\frac{x}{90+t}\frac{\text{lb}}{\text{g}} \times 3\frac{\text{g}}{\text{min}} = \frac{3x}{90+t}\frac{\text{lb}}{\text{min}}.$$

The Initial value problem that gives $x(t)$ is

$$\frac{dx}{dt} = 8 - \frac{3x}{90+t}, \quad x(0) = 90.$$

(3) **Find the general solution of** $(x^2+1)\frac{dy}{dx} + 3xy = 6x$**.**

This is a first order linear problem. Divide both sides by (x^2+1) to obtain

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}.$$

Now the DE is in the form $y' + P(x)y = q(x)$. We multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln x^2+1} = (x^2+1)^{3/2}.$$

Obtain

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{1/2}y = 6x(x^2+1)^{1/2}.$$

Notice that the left side is $\frac{d}{dx}((x^2+1)^{3/2}y)$. We must solve

$$\frac{d}{dx}((x^2+1)^{3/2}y) = 6x(x^2+1)^{1/2}.$$

Integrate both sides with respect to x to obtain

$$(x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + C \quad \text{or}$$

$$\boxed{y = 2 + C(x^2+1)^{-3/2}}.$$

(4) **Find the general solution of** $(x+y)\frac{dy}{dx} = x - y$**.**

This is a homogeneous DE. Divide both sides by x to obtain

$$(1+\frac{y}{x})\frac{dy}{dx} = 1 - \frac{y}{x}.$$

Let $v = \frac{y}{x}$. Multiply both sides by x to obtain $xv = y$. Take the derivative of both sides $x\frac{dv}{dx} + v = \frac{dy}{dx}$. We must solve

$$(1+v)(x\frac{dv}{dx} + v) = 1 - v.$$

We separate the variables:

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$

$$\frac{1+v}{1-2v-v^2}dv = \frac{dx}{x}.$$

Integrate both sides. On the left, let $u = 1 - 2v - v^2$. Observe that $du = -2(1+v)dv$. The equation becomes

$$\frac{-1}{2}\int \frac{du}{u} = \int \frac{dx}{x}.$$

$$\frac{-1}{2}\ln|u| = \ln|x| + C$$
$$\ln|u| = -2\ln|x| - 2C$$

Exponentiate

$$|u| = e^{-2C}|x|^{-2}$$
$$\pm u = e^{-2C}|x|^{-2}$$
$$u = \pm e^{-2C}|x|^{-2}$$

Let K be the constant $\pm e^{-2C}$. Notice that $|x|^{-2} = x^{-2}$.

$$1 - 2v - v^2 = Kx^{-2}$$
$$1 - 2\frac{y}{x} - \frac{y^2}{x^2} = Kx^{-2}$$

Multiply both sides by x^2 to obtain

$$x^2 - 2xy - y^2 = K.$$

(If you got this far, I am perfectly satisfied.)

Use the quadratic formula to solve for y:

$$0 = y^2 + 2xy + K - x^2$$

$$\boxed{y = \frac{-2x \pm \sqrt{(2x)^2 - 4(K - x^2)}}{2}}$$

(5) **Find a particular solution of $y'' - y' - 6y = 2\sin 3x$. (In this problem y is a function of x.)**

We try $y = A\sin 3x + B\cos 3x$. We compute

$$y' = 3A\cos 3x - 3B\sin 3x$$
$$y'' = \; - 9A\sin 3x 9B\cos 3x$$

Plug our candidate into the DE to decide what A and B must be:

$$\left.\begin{cases} -9A\sin 3x - 9B\cos 3x \\ -(-3B\sin 3x + 3A\cos 3x) \\ -6(A\sin 3x + B\cos 3x) \end{cases}\right\} = 2\sin 3x.$$

Clean it up

$$\begin{cases} -15A + 3B = 2 \\ -3A - 15B = 0 \end{cases}$$

The bottom equation tells us that $A = -5B$. The top equation becomes

$$-15(-5B) + 3B = 2.$$

Thus $B = \frac{1}{39}$ and $A = \frac{-5}{39}$. We conclude that

$$\boxed{y = \frac{-5}{39}\sin 3x + \frac{1}{39}\cos 3x}$$

is a particular solution of the DE.

(6) **Find the general solution of $y'' - 4y' + 5y = 0$. (In this problem y is a function of x.)**

We try $y = e^{rx}$ and consider the characteristic equation

$$r^2 - 4r + 5 = 0.$$

Use the quadratic equation

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

The general solution of the DE is

$$\boxed{y = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x}$$

(7) **Find the Laplace transform of $\sin^2 x$.**

Recall that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. We find

$$\mathcal{L}\left(\frac{1}{2}(1 - \cos 2x)\right) = \boxed{\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4}\right)}.$$

(8) **Use Laplace transforms to solve the Initial Value Problem**

$$x'' - x' - 6x = 0, \quad x(0) = 2, \quad x'(0) = 1.$$

In this problem x is a function of t.

Let $X = \mathcal{L}(x)$. We compute

$$\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX - 2$$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s(sX - 2) - 1 = s^2 X - 2s - 1$$

Apply $\mathcal{L}$ to the DE to obtain

$$(s^2 X - 2s - 1) - (sX - 2) - 6X = 0$$

Solve for X:

$$(s^2 - s - 6)X = 2s - 1$$

$$X = \frac{2s-1}{s^2-s-6}$$

Observe that $s^2 - s - 6 = (s-3)(s+2)$. Apply the technique of partial fractions

$$\frac{2s-1}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2}.$$

Multiply both sides by $(s-3)(s+2)$ to obtain

$$2s - 1 = A(s+2) + B(s-3)$$

The above equation hold for all choices of s. When $s = -2$, we learn that $-4 - 1 = B(-5)$ or $B = 1$. When $s = 3$ we learn that $5 = 5A$ or $A = 1$. We conclude that

$$x = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{1}{s-3} + \frac{1}{s+2}\right) = e^{3t} + e^{-2t}.$$

The solution of the IVP is $\boxed{x = e^{3t} + e^{-2t}}$.

(9) **Find the Laplace transform of**

$$f(t) = \begin{cases} t & \textbf{if } 0 \le t \le 1 \\ 0 & \textbf{if } 1 < t. \end{cases}$$

We compute

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} t\, dt.$$

/ We use integration by parts: $\int u\, dv = uv - \int v\, du$. Let

$$u = t \quad \text{and} \quad dv = e^{-st}\, dt.$$

Compute

$$du = dt \quad \text{and} \quad v = \int e^{-st}\, dt = \frac{1}{-s} e^{-st}.$$

It follows that

$$\mathcal{L}(f) = t\, \frac{1}{-s} e^{-st}\Big|_0^1 - \int_0^1 \frac{1}{-s} e^{-st} dt = \frac{1}{-s} e^{-s} - \left(\frac{1}{s^2} e^{-st}\Big|_0^1 \right)$$

$$= \frac{e^{-s}}{-s} - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2} \right) = \boxed{-e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) + \frac{1}{s^2}}.$$

(10) **Solve the Initial Value problem**

$$\tfrac{dx}{dt} = x - 3, \quad x(0) = x_0.$$

Graph the solution of the Initial Value Problem for a few different choices of x_0.

One can graph a few solutions without doing any work. Observe that $x = 3$ is a solution of the DE. If $3 < x_0$ then $\frac{dx}{dt}$ is always positive and the graph of $x(t)$ grows away from $x = 3$. If $x_0 < 3$, then $\frac{dx}{dt}$ is always negative and the graph of $x(t)$ falls away from $x = 3$.

One separates the variables in order to solve the DE $\frac{dx}{dt} = x - 3$:

$$\int \frac{dx}{x-3} = \int dt$$

$$\ln|x-3| = t + C$$

$$|x-3| = e^C e^t$$

$$\pm(x-3) = e^C e^t$$

$$x - 3 = \pm e^C e^t.$$

Let K be the constant $\pm e^C$.

$$x - 3 = Ke^t.$$

We evaluate K by plugging in $x(0) = x_0$:

$$x_0 - 3 = Ke^0.$$

Thus $K = x_0 - 3$ and the solution of the IVP is

$$\boxed{x = 3 + (x_0 - 3)e^t.}$$

The graph is on the next page.

The graph of $X(t) = 3 + (X_0 - 3)e^t$ for various choices of X_0