Problem 37 in Section 7.3. Use Laplace transforms to solve the Initial Value Problem:

$$x'' + 4x' + 13x = te^{-t}, \quad x(0) = 0, \quad x'(0) = 2.$$

Solution. Let $X = \mathcal{L}(x)$. Use the fact

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

twice to calculate

$$\begin{aligned} \mathcal{L}(x') &= s\mathcal{L}(x) - x(0) = sX \\ \mathcal{L}(x'') &= s\mathcal{L}(x') - x'(0) = s^2X - 2 \end{aligned}$$

We know $\mathcal{L}(t) = \frac{1}{s^2}$. It follows that $\mathcal{L}(e^{-t}t) = \frac{1}{(s+1)^2}$. Transform the original Initial Value Problem to

$$\begin{aligned} s^2X - 2 + 4sX + 13X &= \tfrac{1}{(s+1)^2} \\ (s^2 + 4s + 13)X &= 2 + \tfrac{1}{(s+1)^2} \\ X &= \tfrac{2}{(s^2+4s+13)} + \tfrac{1}{(s+1)^2(s^2+4s+13)} \end{aligned} \tag{23}$$

We apply the technique of partial fractions. We find numbers A, B, C, ad D, with

$$\frac{1}{(s+1)^2(s^2+4s+13)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4s+13}.$$

Multiply both sides by $(s+1)^2(s^2+4s+13)$ to obtain

$$1 = A(s+1)(s^2+4s+13) + B(s^2+4s+13) + (Cs+D)(s+1)^2$$

$$1 = \begin{cases} \ \ A(s^3 + 5s^2 + 17s + 13) \\ +B(\qquad s^2 + 4s + 13) \\ +C(s^3 + 2s^2 + s) \\ +D(\qquad s^2 + 2s + 1) \end{cases}$$

$$1 = (A+C)s^3 + (5A+B+2C+D)s^2 + (17A+4B+C+2D) + (13A+13B+D)$$

We want A, B, C, and D with

$$\begin{cases} 0 = A + \qquad C \\ 0 = 5A + B + 2C + D \\ 0 = 17A + 4B + C + 2D \\ 1 = 13A + 13B + \qquad D \end{cases}$$

Replace Equation 2 with Equation 2 minus 5 times Equation 1.
Replace Equation 3 with Equation 3 minus 17 times Equation 1.

Replace Equation 4 with Equation 4 minus 13 times Equation 1.

$$\begin{cases} 0 = A + \quad C \\ 0 = \quad B - 3C + D \\ 0 = \quad 4B - 16C + 2D \\ 1 = \quad 13B - 13C + D \end{cases}$$

Replace Equation 3 with Equation 3 minus 4 times Equation 2.
Replace Equation 4 with Equation 4 minus 13 times Equation 2.

$$\begin{cases} 0 = A + \quad C \\ 0 = \quad B - 3C + D \\ 0 = \quad - 4C - 2D \\ 1 = \quad 26C - 12D \end{cases}$$

Divide Equation 3 by -2.

$$\begin{cases} 0 = A + \quad C \\ 0 = \quad B - 3C + \quad D \\ 0 = \quad 2C + \quad D \\ 1 = \quad 26C - 12D \end{cases}$$

Replace Equation 4 with Equation 4 minus 13 times Equation 3.

$$\begin{cases} 0 = A + \quad C \\ 0 = \quad B - 3C + \quad D \\ 0 = \quad 2C + \quad D \\ 1 = \quad - 25D \end{cases}$$

Thus, $D = \frac{-1}{25}$, $C = \frac{1}{50}$, $B = 3C - D = \frac{5}{50}$, and $A = \frac{-1}{50}$. We resume from (23):

$$X = \frac{2}{(s^2 + 4s + 13)} + \frac{1}{50}\left[\frac{-1}{s+1} + \frac{5}{(s+1)^2} + \frac{s-2}{(s^2 + 4s + 13)}\right]$$

$$X = \frac{1}{50}\left[\frac{-1}{s+1} + \frac{5}{(s+1)^2} + \frac{s-2+100}{(s^2 + 4s + 13)}\right]$$

$$X = \frac{1}{50}\left[\frac{-1}{s+1} + \frac{5}{(s+1)^2} + \frac{(s+2)+96}{(s+2)^2 + 9)}\right]$$

$$X = \frac{1}{50}\left[\frac{-1}{s+1} + \frac{5}{(s+1)^2} + \frac{(s+2)}{(s+2)^2 + 9)} + \frac{96}{(s+2)^2 + 9)}\right]$$

$$X = \frac{1}{50}\left[\frac{-1}{s+1} + \frac{5}{(s+1)^2} + \frac{(s+2)}{(s+2)^2 + 9)} + 32\frac{3}{(s+2)^2 + 9)}\right]$$

Thus,

$$x = \mathcal{L}^{-1}(X) = \frac{1}{50}\Big[-e^{-t} + 5te^{-t} + e^{-2t}\cos 3t + 32e^{-2t}\sin 3t\Big]$$

The solution to the Initial Value Problem is

$$\boxed{x = \frac{1}{50}\Big[-e^{-t} + 5te^{-t} + e^{-2t}\cos 3t + 32e^{-2t}\sin 3t\Big].}$$

Check. Plug

$$\begin{aligned}
x &= \frac{1}{50}\Big(e^{-t}(5t-1) + e^{-2t}(\cos 3t + 32\sin 3t)\Big)\\
x' &= \frac{1}{50}\begin{cases}(e^{-t}5 - e^{-t}(5t-1)\\ +e^{-2t}(-3\sin 3t + 96\cos 3t) - 2e^{-2t}(\cos 3t + 32\sin 3t)\end{cases}\\
&= \frac{1}{50}\Big(e^{-t}(-5t+6) + e^{-2t}(-67\sin 3t + 94\cos 3t)\Big)\\
x'' &= \frac{1}{50}\begin{cases}e^{-t}(-5) - e^{-t}(-5t+6)\\ +e^{-2t}(-3(67)\cos 3t - 3(94)\sin 3t) - 2e^{-2t}(-67\sin 3t + 94\cos 3t)\end{cases}\\
&= \frac{1}{50}\Big(e^{-t}(5t-11) - 389\cos 3t - 148\sin 3t\Big)
\end{aligned}$$

into $x'' + 4x' + 13x$ and obtain

$$\frac{1}{50}\begin{cases}e^{-t}(5t-11) - 389\cos 3t - 148\sin 3t\\ +4\Big(e^{-t}(-5t+6) + e^{-2t}(-67\sin 3t + 94\cos 3t)\Big)\\ +13\Big(e^{-t}(5t-1) + e^{-2t}(\cos 3t + 32\sin 3t)\Big)\end{cases}$$

$$= \frac{1}{50}\left\{\begin{array}{l}e^{-t}\Big(5 - 20 + 65)t + (-11 + 24 - 13)\Big)\\ +\cos 3t(-389 + 4(94) + 12)\\ +\sin 3t(-148 - 4(67) + 13(32))\end{array}\right\} = te^{-t}\checkmark;$$

$x(0) = 0\checkmark$; and $x'(0) = \frac{5+1+96-2}{50} = 2\checkmark$. The proposed answer does everything it is supposed to do. It is correct.