**Problem 33 in Section 7.3.** Use Laplace transforms to solve the Initial Value Problem:

$$x'''' + x = 0, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = 1.$$

(You probably want to use some of the problems 23-26 in 7.3 when you do this problem.)

**Solution.** Let $X = \mathcal{L}(x)$. Use the fact

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

four times to calculate

$$\begin{aligned}
\mathcal{L}(x') &= s\mathcal{L}(x) - x(0) = sX \\
\mathcal{L}(x'') &= s\mathcal{L}(x') - x'(0) = s^2X \\
\mathcal{L}(x''') &= s\mathcal{L}(x'') - x''(0) = s^3X \\
\mathcal{L}(x'''') &= s\mathcal{L}(x''') - x'''(0) = s^4X - 1
\end{aligned}$$

Transform the Differential Equation to obtain

$$s^4X - 1 + X = 0$$

$$(s^4 + 1)X = 1$$

$$X = \frac{1}{s^4+1}$$

$$x = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}(\tfrac{1}{s^4+1})$$

Use problem 26 in Section 7.3:

$$\mathcal{L}^{-1}\left(\frac{1}{s^4+4a^4}\right) = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$$

For us $a = \frac{1}{\sqrt[4]{4}} = \frac{1}{\sqrt{2}}$. Of course,

$$\cosh x = \tfrac{e^x+e^{-x}}{2} \quad \text{and} \quad \sinh x = \tfrac{e^x-e^{-x}}{2}.$$

Thus,

$$x = \mathcal{L}^{-1}(\tfrac{1}{s^4+1}) = \frac{1}{4(\frac{1}{\sqrt{2}})^3}\left[\frac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\sin(\tfrac{t}{\sqrt{2}}) - \frac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\cos(\tfrac{t}{\sqrt{2}})\right].$$

Observe that $4(\frac{1}{\sqrt{2}})^3 = \sqrt{2}$. We conclude that

$$\boxed{x = \frac{1}{\sqrt{2}}\left[\left(\frac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\right)\sin(\tfrac{t}{\sqrt{2}}) - \left(\frac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\cos(\tfrac{t}{\sqrt{2}})\right].}$$

**Check.** Plug

$$x = \tfrac{1}{\sqrt{2}}\Big[\left(\tfrac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\right)\sin(\tfrac{t}{\sqrt{2}}) - \left(\tfrac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\cos(\tfrac{t}{\sqrt{2}})\Big]$$

$$\begin{aligned}
x' &= \frac{1}{\sqrt{2}}\begin{cases}\left(\tfrac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\right)\tfrac{1}{\sqrt{2}}\cos(\tfrac{t}{\sqrt{2}}) - \left(\tfrac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\tfrac{(-1)}{\sqrt{2}}\sin(\tfrac{t}{\sqrt{2}}) \\ +\tfrac{1}{\sqrt{2}}\left(\tfrac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\sin(\tfrac{t}{\sqrt{2}}) - \tfrac{1}{\sqrt{2}}\left(\tfrac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\right)\cos(\tfrac{t}{\sqrt{2}})\end{cases} \\
&= \tfrac{1}{\sqrt{2}}\tfrac{2}{\sqrt{2}}\left(\tfrac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\sin(\tfrac{t}{\sqrt{2}}) \\
&= \tfrac{1}{2}(e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}}) \\
x'' &= \tfrac{1}{2}\Big[(e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\tfrac{1}{\sqrt{2}}\cos(\tfrac{t}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}})\Big] \\
&= \tfrac{1}{2\sqrt{2}}\Big[(e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}}) + (e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}})\Big] \\
x''' &= \tfrac{1}{2\sqrt{2}}\begin{cases}\Big[(e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})(\tfrac{-1}{\sqrt{2}})\sin(\tfrac{t}{\sqrt{2}}) + (e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})(\tfrac{1}{\sqrt{2}})\cos(\tfrac{t}{\sqrt{2}})\Big] \\ (\tfrac{1}{\sqrt{2}})\Big[(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}}) + (e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}})\Big]\end{cases} \\
&= (\tfrac{1}{2\sqrt{2}})(\tfrac{2}{\sqrt{2}})(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}}) \\
&= (\tfrac{1}{2})(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}}) \\
x'''' &= (\tfrac{1}{2})\Big[(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})(-1)\tfrac{1}{\sqrt{2}}\sin(\tfrac{t}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}(e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}})\Big] \\
&= (\tfrac{1}{2\sqrt{2}})\Big[-(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}}) + (e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}})\Big]
\end{aligned}$$

into $x'''' + x$ and get

$$\begin{cases}(\tfrac{1}{2\sqrt{2}})\Big[-(e^{\frac{t}{\sqrt{2}}} + e^{\frac{-t}{\sqrt{2}}})\sin(\tfrac{t}{\sqrt{2}}) + (e^{\frac{t}{\sqrt{2}}} - e^{\frac{-t}{\sqrt{2}}})\cos(\tfrac{t}{\sqrt{2}})\Big] \\ +\tfrac{1}{\sqrt{2}}\Big[\left(\tfrac{e^{\frac{t}{\sqrt{2}}}+e^{\frac{-t}{\sqrt{2}}}}{2}\right)\sin(\tfrac{t}{\sqrt{2}}) - \left(\tfrac{e^{\frac{t}{\sqrt{2}}}-e^{\frac{-t}{\sqrt{2}}}}{2}\right)\cos(\tfrac{t}{\sqrt{2}})\Big],\end{cases}$$

which is $0\checkmark$; $x(0) = 0\checkmark$; $x'(0) = 0\checkmark$; $x''(0) = 0\checkmark$; and $x'''(0) = 1\checkmark$. The proposed answer does everything it is supposed to do; it is correct!