Problem 19 in Section 7.3. Find the inverse Laplace transform of $F(s) = \frac{s^2-2s}{s^4+5s^2+4}$.

Solution. The denominator factors as

$$(s^2+4)(s^2+1).$$

We apply the technique of partial fractions to write

$$\frac{s^2-2s}{s^4+5s^2+4} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}.$$

Multiply both sides by $(s^2+4)(s^2+1)$:

$$s^2-2s = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$$s^2-2s = (A+C)s^3 + (B+D)s^2 + (A+4C)s + B + 4D$$

We want

$$\begin{cases} 0 = & A+C \\ 1 = & B+D \\ -2 = & A+4C \\ 0 = & B+4D \end{cases}$$

Equation 1 gives $A = -C$; Equation 3 gives $-2 = -C + 4C$; so $C = -\frac{2}{3}$ and $A = \frac{2}{3}$. Equation 4 gives $B = -4D$. Equation 2 gives $1 = -4D + D$. It follows that $D = -\frac{1}{3}$ and $B = \frac{4}{3}$. We compute

$$\mathcal{L}^{-1}\left(\frac{s^2-2s}{s^4+5s^2+4}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{2s+4}{s^2+4} + \frac{-2s-1}{s^2+1}\right)$$

$$= \boxed{\frac{1}{3}(2\cos 2t + 2\sin 2t - 2\cos t - \sin t)}.$$