Problem 23 in Section 7.2. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s^2-1)}.$$

Solution. We use $\mathcal{L}^{-1}(\frac{F(s)}{s}) = \int_0^t \mathcal{L}^{-1}(F(s))|_\tau d\tau$ twice. In the course of this calculation we will use

$$\frac{1}{s^2-1} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right].$$

One uses the technique of partial fractions to find this. Find numbers A and B so that

$$\frac{1}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1}.$$

We compute

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2-1)}\right) &= \int_0^t \left[\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)\right]_\tau d\tau \\
&= \int_0^t \int_0^\tau \left[\mathcal{L}^{-1}\left(\frac{1}{(s^2-1)}\right)\right]_\theta d\theta\, d\tau \\
&= \frac{1}{2}\int_0^t \int_0^\tau \left[\mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)\right]_\theta d\theta\, d\tau \\
&= \frac{1}{2}\int_0^t \int_0^\tau (e^\theta - e^{-\theta})\, d\theta\, d\tau \\
&= \frac{1}{2}\int_0^t (e^\theta + e^{-\theta})\Big|_0^\tau d\tau \\
&= \frac{1}{2}\int_0^t (e^\tau + e^{-\tau} - (1+1))d\tau \\
&= \frac{1}{2}(e^\tau - e^{-\tau} - 2\tau\Big|_0^t) \\
&= \frac{1}{2}(e^t - e^{-t} - 2t - (e^0 - e^{-0} - 2(0)) \\
&= \boxed{\frac{1}{2}(e^t - e^{-t} - 2t)}
\end{aligned}$$