Problem 5 in Section 7.1. Use the definition of $\mathcal{L}$ to compute $\mathcal{L}(f(t))$ for $f(t) = \frac{e^t - e^{-t}}{2}$.

Solution. Recall that $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$. We compute

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st}\frac{e^t - e^{-t}}{2}\,dt \\
&= \frac{1}{2}\int_0^\infty (e^{t(1-s)} - e^{t(-1-s)})\,dt \\
&= \frac{1}{2}\lim_{b\to\infty}\Big(\frac{1}{1-s}e^{t(1-s)} - \frac{1}{-1-s}e^{t(-1-s)}\Big)\Big|_0^b \\
&= \frac{1}{2}\Big[\lim_{b\to\infty}\Big(\frac{1}{1-s}e^{b(1-s)} - \frac{1}{-1-s}e^{b(-1-s)}\Big) - \Big(\frac{1}{1-s} - \frac{1}{-1-s}\Big)\Big]
\end{aligned}$$

Assume $1 < s$. In this case, $-1-s < 1-s < 0$ and $\lim_{b\to\infty} e^{b(1-s)} = 0$ and $\lim_{b\to\infty} e^{b(-1-s)} = 0$.

$$\begin{aligned}
&= \frac{1}{2}\Big(-\Big(\frac{1}{1-s} - \frac{1}{-1-s}\Big)\Big) \\
&= \frac{1}{2}\Big(\frac{1}{s-1} - \frac{1}{s+1}\Big) \\
&= \frac{1}{2}\Big(\frac{s+1}{(s+1)(s-1)} - \frac{s-1}{(s+1)(s-1)}\Big) \\
&= \boxed{\frac{1}{s^2-1},\ \text{provided } 1 < s.}
\end{aligned}$$