Problem 2 in Section 7.1. Use the definition of $\mathcal{L}$ to compute $\mathcal{L}(f(t))$ for $f(t) = t^2$.

Solution. Recall that $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$. We must compute

$$\mathcal{L}(t) = \int_0^\infty e^{-st} t^2\, dt.$$

We use integration by parts:

$$\int u\, dv = uv - \int v\, du.$$

Take $u = t^2$ and $dv = e^{-st}dt$. Compute $du = 2tdt$ and $v = \frac{1}{-s}e^{-st}$. It follows that

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\infty e^{-st} t^2\, dt \\
&= \left[uv - \int vdu\right]_0^\infty \\
&= \left[t^2 \tfrac{1}{-s} e^{-st} - \textstyle\int (\tfrac{1}{-s} e^{-st}) 2tdt\right]_0^\infty \\
&= \left[t^2 \tfrac{1}{-s} e^{-st}\right]_0^\infty - \tfrac{2}{-s} \textstyle\int_0^\infty e^{-st} tdt
\end{aligned}$$

Obviously, we can compute $\int_0^\infty e^{-st} tdt$ because we just did it in problem one. But, it would be more clever to observe that this integral is exactly equal to $\mathcal{L}(t)$ and we know from number one (or any list of Laplace transform formulas) that $\mathcal{L}(t) = \frac{1}{s^2}$, provided $0 < s$.

Also, $\int^\infty$ always means plug a number b in for the variable and take the limit as b goes to infinity.

$$= \lim_{b \to \infty} b^2 \tfrac{1}{-s} e^{-sb} - 0^2 \tfrac{1}{-s} e^{-s(0)} + \tfrac{2}{s}(\tfrac{1}{s^2})$$

When one evaluates $\lim_{b\to\infty} \frac{b^2}{e^{sb}}$, where b is a positive constant, one uses the fact that the exponential function overwhelms all polynomial functions; or more precisely, one uses L'Hopital's rule twice, to see that $\lim_{b\to\infty} \frac{b^2}{e^{sb}} = 0$.

$$= \boxed{\frac{2}{s^3}}$$