Problem 6 in Section 3.5. Find a particular solution of

$$2y'' + 4y' + 7y = x^2.$$

Solution. Try $y = Ax^2 + Bx + C$. Plug

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

into $2y'' + 4y' + 7y = x^2$ and obtain

$$2\Big(2A\Big) + 4\Big(2Ax + B\Big) + 7\Big(Ax^2 + Bx + C\Big) = x^2$$

$$7Ax^2 + (8A + 7B)x + 4A + 4B + 7C = x^2$$

So we want

$$\begin{aligned} 1 &= 7A \\ 0 &= 8A + 7B \\ 0 &= 4A + 4B + 7C \end{aligned}$$

So, $A = \frac{1}{7}$, $B = \frac{-8}{49}$, and $\frac{4}{7^3} = C$. Thus,

$$\boxed{y = \tfrac{1}{7^3}(49x^2 - 56x + 4)}$$

is a particular solution of $2y'' + 4y' + 7y = x^2$.

Check. Plug

$$\begin{aligned} y &= \tfrac{1}{7^3}(49x^2 - 56x + 4) \\ y' &= \tfrac{1}{7^3}(2(49)x - 56) \\ y'' &= \tfrac{1}{7^3}(2)(49) \end{aligned}$$

into $2y'' + 4y' + 7y$ and obtain

$$\tfrac{1}{7^3}\Big(2(2(49)) + 4(2(49)x - 56) + 7(49x^2 - 56x + 4)\Big)$$

$$\tfrac{1}{7^3}\Big(\underbrace{[4(49) + 4(-56) + 7(4)]}_{4(49-56+7)=0} + \underbrace{[8(49) - 7(56)]}_{7(56-56)=0}x + 7(49)x^2\Big) = x^2.\checkmark$$