Problem 5 in Section 3.5. Find a particular solution of

$$y'' + y' + y = \sin^2 x.$$

Solution. I suppose we could try

$$y = A\sin^2 x + B\tfrac{d}{dx}(\sin^2 x) + C\tfrac{d^2}{dx^2}(\sin^2 x) + \ldots;$$

but that looks like a mess. Instead, we find a different way to write $\sin^2 x$. Recall that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. (You used to know this. When you integrated $\sin^2 x$ in first semester calculus, you used this identity. Now you can get it from Euler's Identity by using $\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$.)

We must solve

$$y'' + y' + y = \tfrac{1}{2}(1 - \cos 2x).$$

Try $y = A + B\sin 2x + C\cos 2x$. Plug

$$\begin{aligned} y &= A + B\sin 2x + C\cos 2x \\ y' &= 2B\cos 2x - 2C\sin 2x \\ y'' &= -4B\sin 2x - 4C\cos 2x \end{aligned}$$

into

$$y'' + y' + y = \tfrac{1}{2}(1 - \cos 2x)$$

and obtain

$$\left.\begin{cases} -4B\sin 2x - 4C\cos 2x \\ +2B\cos 2x - 2C\sin 2x \\ +A + B\sin 2x + C\cos 2x \end{cases}\right\} = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$$

$$A + \sin 2x(-4B - 2C + B) + \cos 2x(-4C + 2B + C) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$$

So we want

$$\begin{cases} A = \frac{1}{2} \\ -3B - 2C = 0 \\ 2B - 3C = -\frac{1}{2} \end{cases}$$

Add $\frac{2}{3}$ times Equation 2 to Equation 3 and obtain

$$\begin{cases} A = \frac{1}{2} \\ -3B - 2C = 0 \\ -\frac{13}{3}C = -\frac{1}{2} \end{cases}$$

Thus $C = \frac{3}{26}$, $B = -\frac{2}{26}$, and $A = \frac{1}{2}$. We conclude that

$$\boxed{y = \frac{1}{2} - \frac{1}{13}\sin 2x + \frac{3}{26}\cos 2x}$$

is a particular solution of $y'' + y' + y = \frac{1}{2}(1 - \cos 2x)$.

Check. Plug

$$\begin{aligned} y &= \frac{1}{2} - \frac{1}{13}\sin 2x + \frac{3}{26}\cos 2x \\ y' &= -\frac{2}{13}\cos 2x - \frac{3}{13}\sin 2x \\ y'' &= +\frac{4}{13}\sin 2x - \frac{6}{13}\cos 2x \end{aligned}$$

into $y'' + y' + y$ and obtain

$$\begin{aligned} &(+\tfrac{4}{13}\sin 2x - \tfrac{6}{13}\cos 2x) + (-\tfrac{2}{13}\cos 2x - \tfrac{3}{13}\sin 2x) + (\tfrac{1}{2} - \tfrac{1}{13}\sin 2x + \tfrac{3}{26}\cos 2x) \\ &= \tfrac{1}{2} + (+\tfrac{4}{13} - \tfrac{3}{13} - \tfrac{1}{13})\sin 2x + (-\tfrac{6}{13} - \tfrac{2}{13} + \tfrac{3}{26})\cos 2x \\ &= \tfrac{1}{2} + (0)\sin 2x +)\tfrac{-12-4+3}{26})\cos 2x \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x). \checkmark \end{aligned}$$