Problem 43 in Section 3.5.

(a) Find a Trig identity which expresses $\cos^3 x$ in terms of $\cos ax$ for various values of a. (Hint: Use Euler's Identity.)

(b) Find the general solution of

$$y'' + 4y = \cos^3 x.$$

Solution. First we look at $(e^{i\theta})^3 = e^{i(3\theta)}$. The left side is equal to

$$\begin{aligned}
&(\cos\theta + i\sin\theta)^3 \\
&= \begin{cases} \cos^3\theta + i(3\cos^2\theta\sin\theta) \\ +i^2(3\cos\theta\sin^2\theta) + i^3\sin^3\theta \end{cases} \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

The right side is equal to $\cos 3\theta + i\sin 3\theta$ Equate the real part of the left side with the real part of the right side:

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$$

Replace $\sin^2\theta$ with $1 - \cos^2\theta$:

$$\cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = \cos 3\theta$$

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$

Our answer to (a) is

$$\boxed{\cos^3\theta = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)}$$

Now we work on (b). We look for a particular solution for

$$y'' + 4y = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)$$

We try $y = A\cos 3\theta + B\sin 3\theta + C\cos\theta + D\sin\theta$ We plug

$$\begin{aligned}
y &= A\cos 3\theta + B\sin 3\theta + C\cos\theta + D\sin\theta \\
y' &= -3A\sin 3\theta + 3B\cos 3\theta - C\sin\theta + D\cos\theta \\
y'' &= -9A\cos 3\theta - 9B\sin 3\theta - C\cos\theta - D\sin\theta
\end{aligned}$$

into

$$y'' + 4y = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)$$

and obtain

$$\left.\begin{cases}-9A\cos 3\theta - 9B\sin 3\theta - C\cos\theta - D\sin\theta \\ +4\Big(A\cos 3\theta + B\sin 3\theta + C\cos\theta + D\sin\theta)\end{cases}\right\} = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$\left.\begin{cases}(-9A + 4A)\cos 3\theta + (-9B + 4B)\sin 3\theta \\ +(-C + 4C)\cos\theta + (-D + 4D)\sin\theta\end{cases}\right\} = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)$$

We want

$$-5A = \tfrac{1}{4}, \quad -5B = 0, \quad 3C = \tfrac{3}{4}, \quad 3D = 0$$

We take

$$A = -\frac{1}{20}, \quad B = 0, \quad C = \frac{1}{4}, \quad D = 0.$$

We have shown that $y = -\frac{1}{20}\cos 3\theta + \frac{1}{4}\cos\theta$ is a particular solution of the Differential Equation $y'' + 4y = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$.

To find the general solution of the homogeneous problem

$$y'' + 4y = 0,$$

we try $y = e^{rx}$ and consider the characteristic equation $r^2 + 4 = 0$. The roots are $\pm 2i$. The general solution of the homogeneous problem $y'' + 4y = 0$ is $y = c_1\sin 2x + c_2\cos 2x$. The general solution of $y'' + 4y = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ is

$$\boxed{y = c_1\sin 2x + c_2\cos 2x + -\tfrac{1}{20}\cos 3\theta + \tfrac{1}{4}\cos\theta}$$

Check. We plug

$$\begin{aligned} y &= c_1\sin 2x + c_2\cos 2x - \tfrac{1}{20}\cos 3\theta + \tfrac{1}{4}\cos\theta \\ y' &= 2c_1\cos 2x - 2c_2\sin 2x + \tfrac{3}{20}\sin 3\theta - \tfrac{1}{4}\sin\theta \\ y'' &= -4c_1\sin 2x - 4c_2\cos 2x + \tfrac{9}{20}\cos 3\theta - \tfrac{1}{4}\cos\theta \end{aligned}$$

into $y'' + 4y$ and obtain

$$\begin{cases}-4c_1\sin 2x - 4c_2\cos 2x + \tfrac{9}{20}\cos 3\theta - \tfrac{1}{4}\cos\theta \\ +4\Big(c_1\sin 2x + c_2\cos 2x - \tfrac{1}{20}\cos 3\theta + \tfrac{1}{4}\cos\theta\Big)\end{cases}$$

$$\begin{aligned} &= (-4+4)c_1\sin 2\theta + (-4+4)c_2\cos 2\theta + \tfrac{9-4}{20}\cos 3\theta + (-\tfrac{1}{4} + 1)\cos\theta \\ &= \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta \checkmark. \end{aligned}$$