Problem 37 in Section 3.5. Solve the Initial Value Problem

$$y''' - 2y'' + y' = 1 + xe^x, \quad y(0) = y'(0) = 0, \quad y''(0) = 1.$$

Solution.

There are three steps.

First we solve the homogeneous problem.

That is, we find the general solution of $y''' - 2y'' + y' = 0$. We try $y = e^{rx}$. We study the characteristic equation $r^3 - 2r^2 + r = 0$. We study $r(r^2 - 2r + 1) = 0$. We study $r(r-1)^2 = 0$, Three linearly independent solutions of this third order linear homogeneous Differential Equation are $y = 1$, $y = e^x$, and $y = xe^x$. The general solution of $y''' - 2y'' + y' = 0$ is

$$y = c_1 + c_2 e^x + c_3 x e^x.$$

Then we find a particular solution of the given Differential Equation.

We look for a particular solution of $y''' - 2y'' + y' = 1 + xe^x$. Clearly, it is good enough to add a particular solution of $y''' - 2y'' + y' = 1$ to a particular solution of $y''' - 2y'' + y' = xe^x$.

If we are looking for a particular solution of $y''' - 2y'' + y' = 1$, it is pretty clear that $y = x$ works. (If this isn't clear then try $y = A$, notice that it doesn't work, then try $y = Ax + B$ and notice that this does work when $B = 0$ and $A = 1$.) At any rate

$$y = x \text{ is a particular solution of } y''' - 2y' + y' = 1 \tag{16}$$

Now we need a particular solution of $y''' - 2y'' + y' = xe^x$. Normally, we would try $y = Axe^x + Be^x$. But we know that every function of the form $y = Axe^x + Be^x$ is a solution of the homogeneous problem $y''' - 2y'' + y' = 0$; so there is no point trying this candidate. If we were supposed to find a particular solution of $y''' - 2y'' + y' = e^x$, we would try $y = Ax^2e^x$ (because e^x and xe^x both are solutions of $y''' - 2y'' + y' = 0$; so once again, there is no point in trying $y = Ax^2e^x$. So we try $y = Ax^3e^x + Bx^2e^x$. (By the way, if you try a candidate which does not work, you will get a system of equations that has no solution. If this happens to you, it is not a big deal, you just have to start again with a better candidate.)

We plug

$$\begin{aligned}
y &= (Ax^3 + Bx^2)e^x \\
y' &= (Ax^3 + Bx^2)e^x + (3Ax^2 + 2Bx)e^x \\
&= (Ax^3 + (3A + B)x^2 + 2Bx)e^x \\
y'' &= (Ax^3 + (3A + B)x^2 + 2Bx)e^x + (3Ax^2 + (6A + 2B)x + 2B)e^x \\
&= (Ax^3 + (6A + B)x^2 + (6A + 4B)x + 2B)e^x \\
y''' &= (Ax^3 + (6A + B)x^2 + (6A + 4B)x + 2B)e^x + (3Ax^2 + (12A + 2B)x + (6A + 4B))e^x \\
&= (Ax^3 + (9A + B)x^2 + (18A + 6B)x + 6A + 6B)e^x
\end{aligned}$$

into $y''' - 2y'' + y' = xe^x$ and obtain

$$\left.\begin{cases} (Ax^3 + (9A+B)x^2 + (18A+6B)x + 6A + 6B)e^x \\ -2\Big((Ax^3 + (6A+B)x^2 + (6A+4B)x + 2B)e^x\Big) \\ +(Ax^3 + (3A+B)x^2 + 2Bx)e^x \end{cases}\right\} = xe^x.$$

$$\left.\begin{cases} (1-2+1)Ax^3 \\ +\Big((9-12+3)A + (1-2+1)B\Big)x^2 \\ +\Big((18-12)A + (6-8+2)B\Big)x \\ +\Big(6A + (6-4)B\Big)e^x \end{cases}\right\} = xe^x$$

We want to find A and B with

$$(6Ax + 6A + 2B)e^x = xe^x.$$

We want

$$6A = 1 \quad \text{and} \quad 6A + 2B = 0.$$

We take $A = \frac{1}{6}$ and $B = -\frac{1}{2}$.

So

$$y = x + (\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2)e^x$$

is a particular solution of $y''' - 2y'' + y' = 1 + xe^x$. (The "$x$" comes from (16).) The general solution of $y''' - 2y'' + y' = 1 + xe^x$ is

$$y = c_1 + x + (c_2 + c_3x + \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2)e^x.$$

Step 3. We use the initial condition to evaluate the constants.

We compute

$$\begin{aligned} y &= c_1 + x + (c_2 + c_3x + \frac{1}{6}x^3 - \frac{1}{2}x^2)e^x \\ y' &= 1 + (c_2 + c_3x + \frac{1}{6}x^3 - \frac{1}{2}x^2 + c_3 + \frac{1}{2}x^2 - x)e^x \\ &= 1 + (c_2 + c_3x + \frac{1}{6}x^3 + c_3 - x)e^x \\ y'' &= (c_2 + c_3x + \frac{1}{6}x^3 + c_3 - x + c_3 + \frac{1}{2}x^2 - 1)e^x \end{aligned}$$

We use $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$. We must find c_1, c_2, and c_3 so that

$$\begin{cases} 0 = c_1 + c_2 \\ 0 = 1 + c_2 + c_3 \\ 1 = c_2 + 2c_3 - 1 \end{cases}$$

We solve

$$\begin{aligned} 0 &= c_1 + c_2 \\ -1 &= \phantom{c_1 + {}} c_2 + c_3 \\ 2 &= \phantom{c_1 + {}} c_2 + 2c_3 \end{aligned}$$

Replace Equation 3 with Equation 3 minus Equation 2:

$$\begin{aligned} 0 &= c_1 + c_2 \\ -1 &= \phantom{c_1 + {}} c_2 + c_3 \\ 3 &= \phantom{c_1 + c_2 + {}} c_3 \end{aligned}$$

Therefore, $c_3 = 3$, $c_2 = -4$, and $c_1 = 4$.

The solution of the initial value problem

$$y''' - 2y'' + y' = 1 + xe^x, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

is

$$\boxed{y = 4 + x + (-4 + 3x + \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2)e^x.}$$

Check. We plug

$$\begin{aligned} y &= 4 + x + (-4 + 3x + \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2)e^x \\ y' &= 1 + (-4 + 3x + \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2)e^x + (3 + \tfrac{1}{2}x^2 - x)e^x \\ &= 1 + (-1 + 2x + \tfrac{1}{6}x^3)e^x \\ y'' &= (-1 + 2x + \tfrac{1}{6}x^3)e^x + (2 + \tfrac{1}{2}x^2)e^x \\ &= (1 + 2x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3)e^x \\ y''' &= (1 + 2x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3)e^x + (2 + x + \tfrac{1}{2}x^2)e^x \\ &= (3 + 3x + x^2 + \tfrac{1}{6}x^3)e^x \end{aligned}$$

into $y''' - 2y'' + y'$ and obtain

$$\left.\begin{cases} (3 + 3x + x^2 + \tfrac{1}{6}x^3)e^x \\ -2(1 + 2x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3)e^x \\ +1 + (-1 + 2x + \tfrac{1}{6}x^3)e^x \end{cases}\right\} = 1 + xe^x \checkmark;$$

$y(0) = 4 - 4 = 0\checkmark$; $y'(0) = 1 - 1 = 0\checkmark$; and $y''(0) = 1\checkmark$. Our solution does everything it is supposed to do. It is correct.