

Problem 3 in Section 3.5. Find a particular solution of

$$y'' - y' - 6y = 2 \sin 3x.$$

Solution. We try $y = A \sin 3x + B \cos 3x$. Plug

$$\begin{aligned}y &= A \sin 3x + B \cos 3x \\y' &= 3A \cos 3x - 3B \sin 3x \\y'' &= -9A \sin 3x - 9B \cos 3x\end{aligned}$$

into $y'' - y' - 6y = 2 \sin 3x$ and obtain

$$(-9A \sin 3x - 9B \cos 3x) - (3A \cos 3x - 3B \sin 3x) - 6(A \sin 3x + B \cos 3x) = 2 \sin 3x.$$

$$(-9A + 3B - 6A) \sin 3x + (-9B - 3A - 6B) \cos 3x = 2 \sin 3x$$

We want

$$\begin{cases} -15A + 3B = 2 \\ -3A - 15B = 0 \end{cases}$$

I write this equation in the form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and then multiply both sides by the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which is

$$\frac{1}{ad - bc} \begin{bmatrix} -a & c \\ b & -d \end{bmatrix}.$$

$$\begin{bmatrix} -15 & 3 \\ -3 & -15 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Multiply both sides of the equation on the left by

$$\frac{1}{(15)^2 + 9} \begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix} = \frac{1}{234} \begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix}$$

$$\frac{1}{234} \begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} -15 & 3 \\ -3 & -15 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{234} \begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-30}{234} \\ \frac{6}{234} \end{bmatrix} = \begin{bmatrix} \frac{-5}{39} \\ \frac{1}{39} \end{bmatrix}$$

Thus, $y = \frac{1}{39}(-5 \sin 3x + \cos 3x)$ is a particular solution of $y'' - y' - 6y = 2 \sin 3x$.

Check. Plug

$$y = \frac{1}{39}(-5 \sin 3x + \cos 3x)$$

$$y' = \frac{1}{39}(-15 \cos 3x - 3 \sin 3x)$$

$$y'' = \frac{1}{39}(45 \sin 3x - 9 \cos 3x)$$

into $y'' - y' - 6y$ and obtain

$$\frac{1}{39} \left((45 \sin 3x - 9 \cos 3x) - (-15 \cos 3x - 3 \sin 3x) - 6(-5 \sin 3x + \cos 3x) \right)$$

$$\frac{1}{39} \left((45 + 3 + 30) \sin 3x + (-9 + 15 - 6) \cos 3x \right) = 2 \sin 2x. \checkmark$$