Problem 21 in Section 3.4. Solve the Initial Problem

$$x'' + 10x' + 125x = 0, \quad x(0) = 6, \quad x'(0) = 50.$$

Put your answer in the form $x(t) = Ce^{at}\cos(bt - \alpha)$ if this makes sense. Sketch the graph of $x = x(t)$.

Solution. We try $x = e^{rt}$. We must study the characteristic equation

$$r^2 + 10r + 125 = 0$$

We use the quadratic formula: the roots of $ar^2 + br + c = 0$ are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In our problem,

$$\begin{aligned} r &= \frac{-10 \pm \sqrt{100 - 500}}{2} \\ &= \frac{-10 \pm 20i}{2} \\ &= -5 \pm 10i \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= e^{-5t}(c_1 \cos 10t + c_2 \sin 10t) \\ x'(t) &= e^{-5t}(-10c_1 \sin 10t + 10c_2 \cos 10t) - 5e^{-5t}(c_1 \cos 10t + c_2 \sin 10t) \end{aligned}$$

Evaluate $x(t)$ and $x'(t)$ at $t = 0$ to learn that $6 = c_1$ and $50 = 10c_2 - 5c_1$. We conclude that $c_2 = 8$. Thus the solution of the Initial Value Problem is

$$x(t) = e^{-5t}(6\cos 10t + 8\sin 10t).$$

We write our answer in the form $x(t) = Ce^{at}\cos(bt - \alpha)$. Of course,

$$6^2 + 8^2 = 10^2.$$

Divide both sides by $\sqrt{6^2 + 8^2} = 10$:

$$x(t) = 10e^{-5t}(\tfrac{3}{5}\cos 10t + \tfrac{4}{5}\sin 10t).$$

Consider a right triangle with

$$\mathrm{ADJ} = 3, \quad \mathrm{OP} = 4, \quad \text{and} \quad \mathrm{HYP} = 5.$$

Let ϕ be the angle from ADJ to HYP. Observe that $\cos\phi = \frac{3}{5}$ and $\sin\phi = \frac{4}{5}$. (There is a picture of ϕ on the next-to-last page of this solution.)

Use the identity

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

with $\theta = 10t$ and $\phi = \arccos\frac{3}{5}$ to see that

$$\cos(10t - \phi) = \cos 10t \cos\phi + \sin 10t \sin\phi = \tfrac{3}{5}\cos 10t + \tfrac{4}{5}\sin\phi;$$

hence

$$\boxed{x(t) = 10e^{-5t}\cos(10t - \phi), \quad \text{for } \phi = \arccos\tfrac{3}{5}.}$$

Check. Plug

$$\begin{aligned}
x(t) &= 10e^{-5t}\cos(10t - \phi)\\
x'(t) &= 10e^{-5t}(-10)\sin(10t - \phi) - 50e^{-5t}\cos(10t - \phi)\\
&= e^{-5t}\Big(-100\sin(10t - \phi) - 50\cos(10t - \phi)\Big)\\
x''(t) &= \begin{cases} e^{-5t}\Big(-1000\cos(10t - \phi) + 500\sin(10t - \phi)\Big)\\ -5e^{-5t}\Big(-100\sin(10t - \phi) - 50\cos(10t - \phi)\Big)\end{cases}\\
&= e^{-5t}\Big((-1000 + 250)\cos(10t - \phi) + (500 + 500)\sin(10t - \phi)\Big)\\
&= e^{-5t}\Big((-750)\cos(10t - \phi) + (1000)\sin(10t - \phi)\Big)
\end{aligned}$$

into $x'' + 10x' + 125x$ and obtain

$$\begin{cases} +e^{-5t}\Big((-750)\cos(10t - \phi) + (1000)\sin(10t - \phi)\Big)\\ +10\Big[e^{-5t}\Big(-100\sin(10t - \phi) - 50\cos(10t - \phi)\Big)\Big]\\ +125[10e^{-5t}\cos(10t - \phi)]\end{cases}$$

$$= e^{-5t}\Big((-750 - 500 + 1250)\cos(10t - \phi)(1000 - 1000)\sin(10t - \phi) = 0\Big), \checkmark$$

$x(0) = 10\cos\phi = 10(\frac{3}{5}) = 6\checkmark$, and

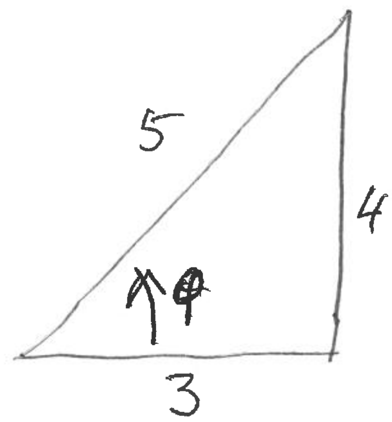
$$\begin{aligned}
x'(0) &= -100\sin(-\phi) - 50\cos(-\phi) = 100\sin\phi - 50\cos\phi\\
&= 100(\tfrac{4}{5}) - 50(\tfrac{3}{5}) = 80 - 30 = 50.\checkmark
\end{aligned}$$

Our solution does everything it is supposed to do. It is correct.

There is a picture of a the angle $\arccos\frac{3}{5}$ on the next page.

There is a sketch of $x(t) = 10e^{-5t}\cos(10t - \phi)$, for $\phi = \arccos\frac{3}{5}$ on the last page of this solution.

The triangle from Section 3.4 number 21



$\cos \varphi = \frac{3}{5}$

$\sin \varphi = \frac{4}{5}$

$\varphi = \arccos\left(\frac{3}{5}\right)$

Picture for 3.4 number 21

We are supposed to draw

$$X = 10e^{-5t} \cos(10t - w)$$

where $w = \arccos(3/5)$

The graph bounces between $X = 10e^{-5t}$ and $X = -10e^{-5t}$