Problem 19 in Section 3.4. Solve the Initial Problem

$$4x'' + 20x' + 169x = 0, \quad x(0) = 4, \quad x'(0) = 16.$$

Put your answer in the form $x(t) = Ce^{at}\cos(bt - \alpha)$ if this makes sense. Sketch the graph of $x = x(t)$.

Solution. We try $x = e^{rt}$. We must study the characteristic equation

$$4r^2 + 20r + 169 = 0$$

We use the quadratic formula: the roots of $ar^2 + br + c = 0$ are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In our problem,

$$\begin{aligned} r &= \frac{-20 \pm \sqrt{400 - 16(169)}}{8} = \frac{-20 \pm \sqrt{16(25 - 169)}}{8} = \frac{-20 \pm 4\sqrt{-144}}{8} \\ &= \frac{-5 \pm 12i}{2} = -\frac{5}{2} \pm 6i. \end{aligned}$$

The general solution of the Differential Equation is

$$x(t) = (c_1 \cos 6t + c_2 \sin 6t)e^{-5t/2}.$$

We use the Initial Conditions to evaluate the constants. We compute

$$x'(t) = -\tfrac{5}{2}(c_1 \cos 6t + c_2 \sin 6t)e^{-5t/2} + (-6c_1 \sin 6t + 6c_2 \cos 6t)e^{-5t/2}.$$

Plug $t = 0$ into $x(t)$ and $x'(t)$ to obtain

$$\begin{aligned} 4 &= x(0) = c_1 \\ 16 &= x'(0) = -\frac{5}{2}c_1 + 6c_2 \end{aligned}$$

or

$$\begin{aligned} 4 &= \quad c_1 \\ 16 &= -\frac{5}{2}c_1 + 6c_2 \end{aligned}$$

Thus, $c_1 = 4$, $c_2 = \frac{13}{3}$, and

$$x(t) = e^{-5t/2}(4\cos 6t + \tfrac{13}{3}\sin 6t). \tag{15}$$

We would like to write $4\cos 6t + \frac{13}{3}\sin 6t$ in the form $C\cos(bt - \alpha)$. We use the identity:

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

We take $\theta = 6t$ and ϕ to be the angle that sits in the right triangle with

$$\text{ADJ} = 4, \quad \text{OP} = \frac{13}{3}, \quad \text{and}$$
$$\text{HYP} = \sqrt{4^2 + \left(\frac{13}{3}\right)^2} = \frac{1}{3}\sqrt{16(9) + 169} = \frac{\sqrt{313}}{3}.$$

A picture of this angle sits on the next-to-last page of this solution. Notice that

$$\cos\phi = \frac{4}{\text{HYP}} = \frac{4}{\frac{\sqrt{313}}{3}} = \frac{12}{\sqrt{313}} \quad \text{and}$$
$$\sin\phi = \frac{\frac{13}{3}}{\text{HYP}}\frac{\frac{13}{3}}{\frac{\sqrt{313}}{3}} = \frac{13}{\sqrt{313}}.$$

Recall our answer from (15):

$$x(t) = e^{-5t/2}(4\cos 6t + \tfrac{13}{3}\sin 6t)$$
$$x(t) = \text{ HYP}\, e^{-5t/2}(\tfrac{4}{\text{HYP}}\cos 6t + \tfrac{\frac{13}{3}}{\text{HYP}}\sin 6t)$$
$$x(t) = \tfrac{\sqrt{313}}{3}e^{-5t/2}(\cos\phi\cos 6t + \sin\phi\sin 6t) \qquad \text{for } \phi = \arccos(\tfrac{12}{\sqrt{313}})$$

$$\boxed{x(t) = \tfrac{\sqrt{313}}{3}e^{-5t/2}(\cos(6t-\phi)), \quad \text{for } \phi = \arccos(\tfrac{12}{\sqrt{313}}).}$$

Check. We plug

$$x(t) = \tfrac{\sqrt{313}}{3}e^{-5t/2}(\cos(6t-\phi))$$
$$x'(t) = \tfrac{\sqrt{313}}{3}\Big(e^{-5t/2}(-6\sin(6t-\phi)) - (\tfrac{5}{2})e^{-5t/2}(\cos(6t-\phi)\Big)$$
$$= \tfrac{\sqrt{313}}{3}e^{-5t/2}\Big(-6\sin(6t-\phi) - \tfrac{5}{2}(\cos(6t-\phi)\Big)$$
$$x''(t) = \tfrac{\sqrt{313}}{3}\begin{cases} e^{-5t/2}\Big(-36\cos(6t-\phi) + 6(\tfrac{5}{2})(\sin(6t-\phi)\Big) \\ -\tfrac{5}{2}e^{-5t/2}\Big(-6\sin(6t-\phi) - \tfrac{5}{2}(\cos(6t-\phi)\Big) \end{cases}$$
$$= \tfrac{\sqrt{313}}{3}e^{-5t/2}\begin{cases} \Big(-36\cos(6t-\phi) + 6(\tfrac{5}{2})(\sin(6t-\phi)\Big) \\ -\tfrac{5}{2}\Big(-6\sin(6t-\phi) - \tfrac{5}{2}(\cos(6t-\phi)\Big) \end{cases}$$
$$= \tfrac{\sqrt{313}}{3}e^{-5t/2}\Big((-36 + \tfrac{25}{4})\cos(6t-\phi) + (15+15)\sin(6t-\phi)$$

into $4x'' + 20x' + 169x$ and obtain

$$\tfrac{\sqrt{313}}{3}e^{-5t/2}\begin{cases} (4(-36 + \tfrac{25}{4}) + 20(-\tfrac{5}{2}) + 169)\cos(6t-\phi) \\ +(4(30) + 20(-6))\sin(6t-\phi) \end{cases} = 0\checkmark,$$

$$x(0) = \frac{\sqrt{313}}{3}\cos(-\phi) = \frac{\sqrt{313}}{3}\cos(\phi) = (\frac{\sqrt{313}}{3})\frac{12}{\sqrt{313}} = 4\checkmark$$

and

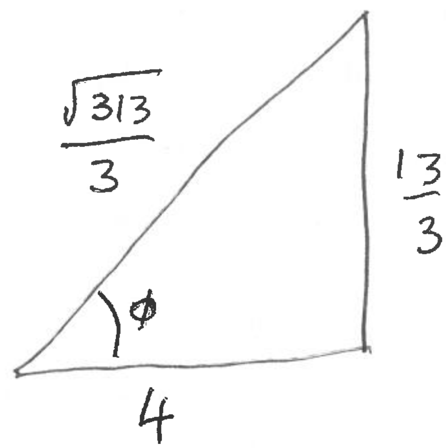
$$\begin{aligned} x'(0) &= \frac{\sqrt{313}}{3}\Big(-6\sin(-\phi) - \frac{5}{2}(\cos(-\phi)\Big) \\ &= \frac{\sqrt{313}}{3}\Big(6\frac{13}{\sqrt{313}} - \frac{5}{2}\frac{12}{\sqrt{313}}\Big) = 26 - 10 = 16.\checkmark \end{aligned}$$

Our proposed answer does everything it is supposed to do. It is correct.

The next page is a picture of the triangle with $\cos\phi = \frac{12}{\sqrt{313}}$.

The last page is a sketch of
$x(t) = \frac{\sqrt{313}}{3}e^{-5t/2}(\cos(6t - \phi)),$ for $\phi = \arccos(\frac{12}{\sqrt{313}})$.

The Triangle for Section 3.4 number 19



$$\cos\phi = \frac{ADJ}{HYP} = \frac{4}{\frac{\sqrt{313}}{3}} = \frac{12}{\sqrt{313}}$$

$$\sin\phi = \frac{OP}{HYP} = \frac{\frac{13}{3}}{\frac{\sqrt{313}}{3}} = \frac{13}{\sqrt{313}}$$

Picture for 3.4 number 19

We are supposed to draw

$$x = \frac{\sqrt{313}}{3} e^{-\frac{5}{2}t} \cos(6t - w) \quad \text{where } w = \arccos \frac{12}{\sqrt{313}}$$

The graph bounces between $x = \frac{\sqrt{313}}{3} e^{-\frac{5}{2}t}$ and $x = -\frac{\sqrt{313}}{3} e^{-\frac{5}{2}t}$