Problem 5 in Section 3.3. Find the general solution of $y'' + 6y' + 9y = 0$.

Solution. We try $y = e^{rx}$. We plug y, $y' = re^{rx}$ and $y'' = r^2e^{rx}$ into the Differential Equation. We want

$$r^2e^{rx} + 6re^{rx} + 9e^{rx} = 0.$$

We want $e^{rx}(r^2 + 6r + 9) = 0$. If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want $r^2 + 6r + 9 = 0$. We want $(r + 3)^2 = 0$. It follows that $y = e^{-3x}$ and $y = xe^{-3x}$ are solutions of the given linear homogeneous Differential Equation with constant coefficients. The general solution of $y'' + 6y' + 9y = 0$ is $\boxed{y = c_1e^{-3x} + c_2xe^{-3x}}$.

Check. We plug

$$\begin{aligned} y &= c_1e^{-3x} + c_2xe^{-3x} \\ y' &= -3c_1e^{-3x} + c_2e^{-3x} - 3c_2xe^{-3x} \\ &= (-3c_1 + c_2)e^{-3x} - 3c_2xe^{-3x} \\ y'' &= -3(-3c_1 + c_2)e^{-3x} - 3c_2e^{-3x} + 9c_2xe^{-3x} \\ &= (9c_1 - 6c_2)e^{-3x} + 9c_2xe^{-3x} \end{aligned}$$

into $y'' + 6y' + 9y$ and obtain

$$\begin{cases} \Big((9c_1 - 6c_2)e^{-3x} + 9c_2xe^{-3x}\Big) \\ +6\Big((-3c_1 + c_2)e^{-3x} - 3c_2xe^{-3x}\Big) \\ +9\Big(c_1e^{-3x} + c_2xe^{-3x}\Big) \end{cases}$$

$$= [(9 - 18 + 9)c_1 + (-6 + 6)c_2]e^{-3x} + (9 - 18 + 9)c_2xe^{-3x} = 0.$$