Problem 49 in Section 3.3. Solve the Initial Problem

$$y'''' = y''' + y'' + y' + 2y, \quad y(0) = 0, y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 30$$

Hint: one solution is $y = \cos(2x)$.

Solution. Write the Differential Equation in the form

$$y'''' - y''' - y'' - y' - 2y.$$

We try $y = e^{rx}$. We plug y, $y' = re^{rx}$, $y'' = r^2e^{rx}$, $y''' = r^3e^{rx}$, and $y'''' = r^4e^{rx}$ into the Differential Equation. We want

$$r^4e^{rx} - r^3e^{rx} - r^2e^{rx} - re^{rx} - 2e^{rx} = 0.$$

We want

$$e^{rx}(r^4 - r^3 - r^2 - r - 2) = 0.$$

If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want

$$r^4 - r^3 - r^2 - r - 2 = 0.$$

If a polynomial with integer coefficients has a rational root, then that rational root is a factor of the constant term divided by a factor of the leading coefficient. In particular, if the above polynomial has any rational roots, then these roots are a factor of 2 divided by a factor of 1. In other words, the only possible rational roots of our polynomial are ± 1 or ± 2. Lets, see if any of those four numbers is a root of our polynomial.

Is $r = 1$ a root of our polynomial? Is $1^4 - 1^3 - 1^2 - 1 - 2 = 0$? No. So we try another one.

Is $r = -1$ a root of our polynomial? Is $(-1)^4 - (-1)^3 - (-1)^2 - (-1) - 2 = 0$? Is $1 + 1 - 1 + 1 - 2 = 0$? Yes! So $r + 1$ is a factor of $r^4 - r^3 - r^2 - r - 2$. To find the other factor, do long division (or do it in your head). At any rate,

$$r^4 - r^3 - r^2 - r - 2 = (r + 1)(r^3 - 2r^2 + r - 2).$$

We still want to factor $r^3 - 2r^2 + r - 2$. The only possible rational roots are ± 1 or ± 2. It turns out that neither 1 nor -1 is a root; however, 2 is a root and

$$(r^3 - 2r^2 + r - 2) = (r - 2)(r^2 + 1).$$

It follows that

$$r^4 - r^3 - r^2 - r - 2 = (r + 1)(r - 2)(r^2 + 1).$$

The roots of $r^4 - r^3 - r^2 - r - 2$ are $-1, 2, i, -i$ and e^{-x}, e^{2x}, $\cos x$ and $\sin x$ are four linearly independent functions which are solutions of the homogeneous linear Differential Equation. The general solution of the Differential equation is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x.$$

Now we evaluate the constants. Take the derivatives:

$$\begin{aligned} y &= c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x \\ y' &= -c_1 e^{-x} + 2c_2 e^{2x} - c_3 \sin x + c_4 \cos x \\ y'' &= c_1 e^{-x} + 4c_2 e^{2x} - c_3 \cos x - c_4 \sin x \\ y''' &= -c_1 e^{-x} + 8c_2 e^{2x} + c_3 \sin x - c_4 \cos x \end{aligned}$$

We want

$$\begin{aligned} 0 = y(0) &= c_1 e^{-(0)} + c_2 e^{2(0)} + c_3 \cos(0) + c_4 \sin(0) \\ 0 = y'(0) &= -c_1 e^{-(0)} + 2c_2 e^{2(0)} - c_3 \sin(0) + c_4 \cos(0) \\ 0 = y''(0) &= c_1 e^{-(0)} + 4c_2 e^{2(0)} - c_3 \cos(0) - c_4 \sin(0) \\ 30 = y'''(0) &= -c_1 e^{-(0)} + 8c_2 e^{2(0)} + c_3 \sin(0) - c_4 \cos(0) \end{aligned}$$

We want

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= -c_1 + 2c_2 + c_4 \\ 0 &= c_1 + 4c_2 - c_3 \\ 30 &= -c_1 + 8c_2 - c_4 \end{aligned}$$

Replace Equation 2 with Equation 2 plus Equation 1.
Replace Equation 3 with Equation 3 minus Equation 1.
Replace Equation 4 with Equation 4 plus Equation 1.

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= +3c_2 + c_3 + c_4 \\ 0 &= +3c_2 - 2c_3 \\ 30 &= +9c_2 + c_3 - c_4 \end{aligned}$$

Replace Equation 3 with Equation 3 minus Equation 2.
Replace Equation 4 with Equation 4 minus 3 times Equation 2.

$$\begin{aligned}
0 &= c_1 + \ \ c_2 + c_3 \\
0 &= \quad + 3c_2 + c_3 + c_4 \\
0 &= \qquad\qquad - 3c_3 - c_4 \\
30 &= \qquad\qquad - 2c_3 - 4c_4
\end{aligned}$$

Replace Equation 4 with Equation 4 minus $\frac{2}{3}$ times Equation 3

$$\begin{aligned}
0 &= c_1 + \ \ c_2 + c_3 \\
0 &= \quad + 3c_2 + c_3 + c_4 \\
0 &= \qquad\qquad - 3c_3 - c_4 \\
30 &= \qquad\qquad\qquad - \tfrac{10}{3}c_4
\end{aligned}$$

Thus, $c_4 = 30(-\frac{3}{10}) = -9$, $c_3 = 3$, $c_2 = 2$, and $c_1 = -5$. The solution of the Initial Value Problem is

$$\boxed{y = -5e^{-x} + 2e^{2x} + 3\cos x - 9\sin x.}$$

Check. Plug the derivatives

$$\begin{aligned}
y &= \ -5e^{-x} + 2e^{2x} + 3\cos x - 9\sin x \\
y' &= 5e^{-x} + 4e^{2x} - 3\sin x - 9\cos x \\
y'' &= \ -5e^{-x} + 8e^{2x} - 3\cos x + 9\sin x \\
y''' &= 5e^{-x} + 16e^{2x} + 3\sin x + 9\cos x \\
y'''' &= \ -5e^{-x} + 32e^{2x} + 3\cos x - 9\sin x
\end{aligned}$$

into $y'''' - y''' - y'' - y' - 2y$ and obtain

$$\begin{cases}
\ \ \Big(-5e^{-x} + 32e^{2x} + 3\cos x - 9\sin x\Big) \\
-\Big(5e^{-x} + 16e^{2x} + 3\sin x + 9\cos x\Big) \\
-\Big(-5e^{-x} + 8e^{2x} - 3\cos x + 9\sin x\Big) \\
-\Big(5e^{-x} + 4e^{2x} - 3\sin x - 9\cos x\Big) \\
-2\Big(-5e^{-x} + 2e^{2x} + 3\cos x - 9\sin x\Big)
\end{cases}$$

$$= \begin{cases}
(-5 - 5 + 5 - 5 + 10)e^{-x} \\
+(32 - 16 - 8 - 4 - 4)e^{2x} \\
+(3 - 9 + 3 + 9 - 6)\cos x \\
+(-9 - 3 - 9 + 3 + 18)\sin x
\end{cases}$$

This sum is zero. ✓

We also see that

$$\begin{aligned} y(0) &= -5+2+3=0\checkmark \\ y'(0) &= 5+4-9=0\checkmark \\ y''(0) &= -5+8-3=0\checkmark \\ y'''(0) &= 5+16+9=30\checkmark \end{aligned}$$

Our proposed solution does everything that it is supposed to do. It is correct.