Problem 13 in Section 3.3. Find the general solution of $9y'''+12y''+4y'=0$.

Solution. We try $y=e^{rx}$. We plug y, $y'=re^{rx}$, $y''=r^2e^{rx}$, and $y'''=r^3e^{rx}$ into the Differential Equation. We want

$$9r^3e^{rx}+12r^2e^{rx}+4re^{rx}=0.$$

We want $e^{rx}(9r^3+12r^2+4r)=0$. If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want

$$(9r^3+12r^2+4r)=0$$

$$r(9r^2+12r+4)=0$$

$$r(3r+2)^2=0$$

So, $r=0$ or $r=\frac{-2}{3}$. The root $=\frac{-2}{3}$ has multiplicity two. The general solution of the Differential Equation is

$$\boxed{y=c_1+c_2e^{-\frac{2}{3}x}+c_3xe^{-\frac{2}{3}x}.}$$

Check. We plug

$$\begin{aligned}
y &= c_1+c_2e^{-\frac{2}{3}x}+c_3xe^{-\frac{2}{3}x}\\
y' &= -\tfrac{2}{3}c_2e^{-\frac{2}{3}x}+c_3e^{-\frac{2}{3}x}-\tfrac{2}{3}c_3xe^{-\frac{2}{3}x}\\
&= (-\tfrac{2}{3}c_2+c_3)e^{-\frac{2}{3}x}-\tfrac{2}{3}c_3xe^{-\frac{2}{3}x}\\
y'' &= -\tfrac{2}{3}(-\tfrac{2}{3}c_2+c_3)e^{-\frac{2}{3}x}-\tfrac{2}{3}c_3e^{-\frac{2}{3}x}+\tfrac{4}{9}c_3xe^{-\frac{2}{3}x}\\
&= (\tfrac{4}{9}c_2-\tfrac{4}{3}c_3)e^{-\frac{2}{3}x}+\tfrac{4}{9}c_3xe^{-\frac{2}{3}x}\\
y''' &= -\tfrac{2}{3}(\tfrac{4}{9}c_2-\tfrac{4}{3}c_3)e^{-\frac{2}{3}x}+\tfrac{4}{9}c_3e^{-\frac{2}{3}x}-\tfrac{8}{27}c_3xe^{-\frac{2}{3}x}\\
&= (-\tfrac{8}{27}c_2+\tfrac{12}{9}c_3)e^{-\frac{2}{3}x}-\tfrac{8}{27}c_3xe^{-\frac{2}{3}x}
\end{aligned}$$

into $9y'''+12y''+4y'$ and obtain

$$\begin{cases}
9\Big((-\frac{8}{27}c_2+\frac{12}{9}c_3)e^{-\frac{2}{3}x}-\frac{8}{27}c_3xe^{-\frac{2}{3}x}\Big)\\
+12\Big((\frac{4}{9}c_2-\frac{4}{3}c_3)e^{-\frac{2}{3}x}+\frac{4}{9}c_3xe^{-\frac{2}{3}x}\Big)\\
+4\Big((-\frac{2}{3}c_2+c_3)e^{-\frac{2}{3}x}-\frac{2}{3}c_3xe^{-\frac{2}{3}x}\Big)
\end{cases}$$

$$=(-\tfrac{72}{27}+\tfrac{48}{9}-\tfrac{8}{3})c_2e^{-\frac{2}{3}}+(12-16+4)c_3e^{-\frac{2}{3}}+(-\tfrac{8}{3}+\tfrac{16}{3}-\tfrac{8}{3})c_3xe^{-\frac{2}{3}x}$$

and this is zero because $(-\frac{72}{27}+\frac{48}{9}-\frac{8}{3})=(-\frac{8}{3}+\frac{16}{3}-\frac{8}{3})=0$. Our proposed answer works.