Problem 29 in Section 3.1. Show that $y_1 = x^2$ and $y_2 = x^3$ are both solutions of the Initial Value Problem

$$x^2y'' - 4xy' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Why doesn't the Existence and Uniqueness Theorem apply to this problem?

Solution. We answer the question first? The Existence and Uniqueness Theorem states that if $P_1(x)$, $P_2(x)$, and $Q(x)$ are all continuous on some interval I which contains a, then the Initial Value Problem

$$y'' + P_1(x)y' + P_0(x)y = Q(x), \quad y(a) = b_0, \quad y'(a) = b_1$$

has a unique solution which is defined on all of I. To put $x^2y'' - 4xy' + 6y = 0$ in the proper form, one must divide by x^2:

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$$

So $P_1(x) = -\frac{4}{x}$, $P_2(x) = \frac{6}{x^2}$, and $Q(x) = 0$. Observe that neither $P_1(x)$ nor $P_2(x)$ is continuous at $x = 0$. So the Existence and Uniqueness Theorem tells us nothing about the Initial Value Problem.

We check that $y_1 = x^2$ is a solution of the IVP. Plug $y_1 = x^2$, $y_1' = 2x$, $y_1'' = 2$ into $x^2y'' - 4xy' + 6y$ to obtain

$$x^2(2) - 4x(2x) + 6(x^2) = x^2(2 - 8 + 6) = 0.\checkmark$$

Calculate $y_1(0) = 0^2 = 0\checkmark$ and $y_1'(0) = 2(0) = 0\checkmark$.

We check that $y_2 = x^3$ is a solution of the IVP. Plug $y_2 = x^3$, $y_2' = 3x^2$, $y_2'' = 6x$ into $x^2y'' - 4xy' + 6y$ to obtain

$$x^2(6x) - 4x(3x^2) + 6(x^3) = x^3(6 - 12 + 6) = x^3(0) = 0.\checkmark$$

Calculate $y_2(0) = 0^3 = 0\checkmark$ and $y_2'(0) = 3(0)^2 = 0\checkmark$.