Problem 2 in Section 3.1.

(a) Verify that $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ both are solutions of the Differential Equation $y'' - 9y = 0$.

(b) Solve the Initial Value Problem:

$$y'' - 9y = 0, \quad y(0) = -1, \quad y'(0) = 15.$$

Solution (a) We compute $y_1' = 3e^{3x}$ and $y_1'' = 9e^{3x}$. Thus,

$$y_1'' - 9y = 9e^{3x} - 9e^{3x} = 0,$$

as expected. In a similar manner, we compute $y_2' = -3e^{-3x}$ and $y_2'' = 9e^{-3x}$. Thus,

$$y_2'' - 9y_2 = 9e^{-3x} - 9e^{-3x} = 0,$$

as expected. This completes (a).

(b) We know from (a) that

$$y = c_1e^{3x} + c_2e^{-3x}$$

is the general solution of the Differential Equation. Now we find the constants that allow y to satisfy the Initial Conditions.

Plug $x = 0$ into the equations

$$\begin{aligned} y &= c_1e^{3x} + c_2e^{-3x} \\ y' &= 3c_1e^{3x} - 3c_2e^{-3x} \end{aligned}$$

to learn that

$$\begin{aligned} -1 &= c_1 + c_2 \\ 15 &= 3c_1 - 3c_2 \end{aligned}$$

The solution set is unchanged if we replace equation 2 with equation 2 minus 3 equation 1:

$$\begin{aligned} -1 &= c_1 + c_2 \\ 18 &= \quad - 6c_2 \end{aligned}$$

So, $c_2 = -3$ and $c_1 = 2$ Our answer is $\boxed{y = 2e^{3x} - 3e^{-3x}}$.

Check. We compute $y' = 6e^{3x} + 9e^{-3x}$ and $y'' = 18e^{3x} - 27e^{-3x}$. We plug y, y', and y'' into $y'' - 9y$ and get

$$18e^{3x} - 27e^{-3x} - 9(2e^{3x} - 3e^{-3x})$$

and this is zero. ✓We also evaluate

$$y(0) = 2e^0 - 3e^0 = -1 \checkmark \quad \text{and} \quad y'(0) = 6e^0 + 9e^0 = 15. \checkmark$$

Our answer does everything it is supposed to do. It is correct.