Problem 19 in Section 3.1. Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are both solutions of $yy'' + (y')^2 = 0$, but the sum $y_1 + y_2 = 1 + \sqrt{x}$ is not a solution of $yy'' + (y')^2 = 0$.

The point of this problem is that our tricks for linear Differential Equations do not work for non-linear Differential Equations. In particular if y_1 and y_2 both are solutions of a homogeneous linear Differential Equation, then y_1+y_2 is also a solution of the Differential Equation. This statement is not true for non-linear Differential Equations.

Solution. Plug $y_1 = 1$, $y_1' = 0$, and $y_1'' = 0$ into $yy'' + (y')^2$ and obtain $1(0) + 0^2 = 0$, as expected.

Plug $y_2 = x^{1/2}$, $y' = (1/2)x^{-1/2}$, and $y'' = (-1/4)x^{-3/2}$ into $yy'' + (y')^2$ and obtain

$$\begin{aligned} &x^{1/2}(-1/4)x^{-3/2} + \Big((1/2)x^{-1/2}\Big)^2 \\ &= (-1/4)x^{-1} + (1/4)x^{-1} = 0, \end{aligned}$$

as expected.

On the other hand, when we plug $y_1 + y_2 = (1 + \sqrt{x})$, $(y_1 + y_2)' = (1/2)x^{-1/2}$, and $(y_1 + y_2)'' = (-1/4)x^{-3/2}$ into $yy'' + (y')^2$, we obtain

$$\begin{aligned} &(1 + \sqrt{x})(-1/4)x^{-3/2} + ((1/2)x^{-1/2})^2 \\ =&(-1/4)x^{-3/2} - (1/4)x^{-1} + (1/4)x^{-1} \\ =&(-1/4)x^{-3/2}. \end{aligned}$$

This is not zero. $y_1 + y_2$ is not a solution of $yy'' + (y')^2 = 0$.