

Problem 18 in Section 3.1. Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that $y = cx^3$ is not a solution of $yy'' = 6x^4$, unless c happens to be 1 or -1 .

The point of this problem is that our tricks for linear Differential Equations do not work for non-linear Differential Equations. In particular if y_1 is a solution of a homogeneous linear Differential Equation, then cy_1 is also a solution of the Differential Equation. This statement is not true for non-linear Differential Equations.

Solution. Start with $y = x^3$. Compute $y' = 3x^2$ and $y'' = 6x$. Plug y and y'' into yy'' to obtain

$$x^3(6x) = 6x^4,$$

as expected.

On the other hand, we we plug $y = cx^3$, $y' = 3cx^2$ and $y'' = 6cx$ into yy'' , we obtain

$$cx^3(6cx) = 6c^2x^4,$$

and $6c^2x^4$ is equal to $6x^4$ only when $c^2 = 1$.