Problem 9 in Section 2.2. Consider the Differential Equation $\frac{dx}{dt} = x^2 - 5x + 4$.

- What are the equilibrium solutions of this autonomous Differential Equation?
- Draw the phase diagram for this Differential Equation.
- Are the equilibrium solutions of this Differential Equation stable or unstable?
- Sketch the solution of the Initial Value problem

$$\frac{dx}{dt} = x^2 - 5x + 4 \quad x(0) = x_0, \tag{11}$$

  for a few choices of x_0.
- Solve the Initial Value Problem (11). (Be sure that your answer is in the form x is equal to some function of t.)

We see that $x^2 - 5x + 4 = (x-1)(x-4)$. Thus, $x^2 - 5x + 4$ is equal to zero when $x = 1$ or $x = 4$. We conclude that $x = 1$ and $x = 4$ are the equilibrium solutions of the given Differential Equation. We see that

- if x is greater than four, then $(x-1)(x-4)$ is positive;
- if x is between 1 and 4, then $(x-1)(x-4)$ is negative; and
- if x is less than 1, then $(x-1)(x-4)$ is positive.

(The phase diagram is on the last page of this solution.) We conclude that

- if $4 < x_0$, then the graph of $x = x(t)$ increases away from 4 as t goes to infinity;
- if $1 \le x \le 4$, then the graph of $x = x(t)$ decreases away from 4 and towards 1 as t goes to infinity; and
- if $x_0 < 1$, then the graph of $x = x(t)$ increases towards to 1 as t goes to infinity.

In other words,

The function $x = 4$ is an unstable equilibrium solution of $\frac{dx}{dt} = x^2 - 5x + 4$ and $x = 1$ is a stable equilibrium solution.

On the last page of this solution, we sketched five solutions of the Initial Value Problem (11). In one solution x_0 is bigger than 4; in one solution x_0 is equal to 4; in one solution x_0 is between 1 and 4; in one solution x_0 is equal to 1; and in one solution x is less than 1.

We solve $\frac{dx}{dt} = x^2 - 5x + 4$ by separating the variables and integrating:

$$\int \frac{dx}{x^2 - 5x + 4} = \int dt \tag{12}$$

Factor the denominator $x^2 - 5x + 4 = (x-4)(x-1)$. Use the method of partial fractions to write

$$\frac{1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}.$$

Multiply both sides by $(x-4)(x-1)$:

$$1 = A(x-1) + B(x-4).$$

Plug in $x = 1$ to learn $B = \frac{-1}{3}$. Plug in $x = 4$ to learn $A = \frac{1}{3}$. Use

$$\frac{1}{(x-4)(x-1)} = \frac{1}{3}\left(\frac{1}{x-4} - \frac{1}{x-1}\right)$$

to compute (12):

$$\frac{1}{3}\int \left(\frac{1}{x-4} - \frac{1}{x-1}\right) dx = t + C$$

$$\frac{1}{3}\left(\ln|x-4| - \ln|x-1|\right) = t + C$$

$$\left(\ln|x-4| - \ln|x-1|\right) = 3t + 3C$$

Exponentiate:

$$\frac{|x-4|}{|x-1|} = e^C e^{3t}$$

$$\frac{x-4}{x-1} = \pm e^C e^{3t}$$

Let K be the constant $\pm e^C$.

$$\frac{x-4}{x-1} = Ke^{3t} \tag{13}$$

Plug $t = 0$ in order to learn that

$$\frac{x_0 - 4}{x_0 - 1} = K \tag{14}$$

We continue to write K. We will insert (14) into our final answer. Multiply both sides of (13) by $x - 1$:

$$x - 4 = Ke^{3t}(x-1)$$

Subtract $Ke^{3t}x$ from both sides and add 4 to both sides:

$$x - Ke^{3t}x = 4 - Ke^{3t}.$$

Factor 3 for each term on the left; and divide both sides by $1 - Ke^{3t}$:

$$x = \frac{4 - Ke^{3t}}{1 - Ke^{3t}}.$$

Insert (14):

$$x = \frac{4 - \left(\frac{x_0-4}{x_0-1}\right)e^{3t}}{1 - \left(\frac{x_0-4}{x_0-1}\right)e^{3t}}.$$

$$\boxed{x = \frac{4(x_0-1) - (x_0-4)e^{3t}}{(x_0-1) - (x_0-4)e^{3t}}}.$$

Check! We plug the proposed answer into $\frac{dx}{dt} - x^2 + 5x - 4$. We hope that the result simplifies to become zero. At any rate,

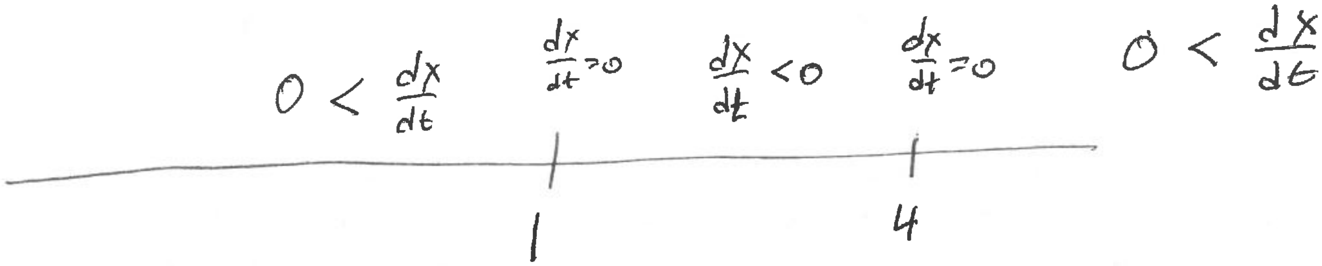
$$\begin{aligned}
&\frac{dx}{dt} - x^2 + 5x - 4 \\
&= \left\{\begin{array}{l}
+(4(x_0-1)-(x_0-4)e^{3t})(-1)((x_0-1)-(x_0-4)e^{3t})^{-2}(-3)(x_0-4)e^{3t} \\
+((x_0-1)-(x_0-4)e^{3t})^{-1}(-3(x_0-4)e^{3t}) \\
-\left(\frac{4(x_0-1)-(x_0-4)e^{3t}}{(x_0-1)-(x_0-4)e^{3t}}\right)^2 \\
+5\frac{4(x_0-1)-(x_0-4)e^{3t}}{(x_0-1)-(x_0-4)e^{3t}} \\
-4
\end{array}\right. \\
&= \frac{\left\{\begin{array}{l}
+(4(x_0-1)-(x_0-4)e^{3t})(-1)(-3)(x_0-4)e^{3t} \\
+((x_0-1)-(x_0-4)e^{3t})(-3(x_0-4)e^{3t}) \\
-\left(4(x_0-1)-(x_0-4)e^{3t}\right)^2 \\
+5(4(x_0-1)-(x_0-4)e^{3t})((x_0-1)-(x_0-4)e^{3t}) \\
-4((x_0-1)-(x_0-4)e^{3t})^2
\end{array}\right.}{((x_0-1)-(x_0-4)e^{3t})^2} \\
&= \frac{\left\{\begin{array}{l}
+12(x_0-1)(x_0-4)e^{3t} - 3(x_0-4)^2e^{6t} \\
-3(x_0-1)(x_0-4)e^{3t} + 3(x_0-4)^2e^{6t} \\
-16(x_0-1)^2 + 8(x_0-1)(x_0-4)e^{3t} - (x_0-4)^2e^{3t} \\
+20(x_0-1)^2 - 20(x_0-1)(x_0-4)e^{3t} - 5(x_0-1)(x_0-4)e^{3t} + 5(x_0-4)^2e^{6t} \\
-4(x_0-1)^2 + 8(x_0-1)(x_0-4)e^{3t} - 4(x_0-4)^2e^{6t}
\end{array}\right.}{((x_0-1)-(x_0-4)e^{3t})^2}
\end{aligned}$$

The numerator is zero. The proposed answer is correct.

The pictures are on the next page.

Pictures for Problem 9 in Section 2.2

Phase Diagram for $\frac{dx}{dt} = x^2 - 5x + 4$



The solution of $\frac{dx}{dt} = x^2 - 5x + 4 \quad x(0) = x_0$

for a few choices of x_0