Problem 5 in Section 2.2. Consider the Differential Equation $\frac{dx}{dt} = x^2 - 4$.

- What are the equilibrium solutions of this autonomous Differential Equation?
- Draw the phase diagram for this Differential Equation.
- Are the equilibrium solutions of this Differential Equation stable or unstable?
- Sketch the solution of the Initial Value problem

  $$\frac{dx}{dt} = x^2 - 4 \quad x(0) = x_0, \tag{6}$$

  for a few choices of x_0.
- Solve the Initial Value Problem (6). (Be sure that your answer is in the form x is equal to some function of t.)

We see that $x^2 - 4 = (x-2)(x+2)$; thus $x^2 - 4$ is equal to zero when $x = 2$ or $x = -2$. We conclude that $x = 2$ and $x = -2$ are equilibrium solutions of the given Differential Equation. We see that

- $(x-2)(x+2)$ is positive for $2 < x$;
- $(x-2)(x+2)$ is negative for $-2 < x < 2$; and
- $(x-2)(x+2)$ is positive for $x < -2$.

(The phase diagram is on the last page of this solution.) We conclude that

- if $2 < x_0$, then the graph of $x = x(t)$ increases away from 2 as t goes to ∞;
- if $-2 < x_0 < 2$, then the graph of $x = x(t)$ decreases away from 2 and towards -2 as t goes to ∞; and
- if $x_0 < -2$, then the graph of $x = x(t)$ increases towards -2 as t goes to ∞.

In other words,

The function $x = 2$ is an unstable equilibrium solution of $\frac{dx}{dt} = x^2 - 4$ and the function $x = -2$ is a stable equilibrium solution of $\frac{dx}{dt} = x^2 - 4$.

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On the last page of this solution, we sketched five solutions of the Initial Value Problem

$$\frac{dx}{dt} = x^2 - 4 \quad x(0) = x_0.$$

In one solution x_0 is bigger than 2; in one solution x_0 is equal to 2; in one solution x_0 is between -2 and 2; in one solution x_0 is equal to -2; and in one solution x_0 is less than -2.

We solve $\frac{dx}{dt} = x^2 - 4$ by separating the variables and integrating:

$$\int \frac{dx}{x^2-4} = \int dt. \tag{7}$$

Use the method of partial fractions:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiply both sides by $(x-2)(x+2)$:

$$1 = A(x+2) + B(x-2).$$

Plug in $x = 2$ to see that $A = \frac{1}{4}$. Plug in $x = -2$ to see that $B = -\frac{1}{4}$. Thus,

$$\frac{1}{(x-2)(x+2)} = \frac{1}{4}\Big(\frac{1}{x-2} - \frac{1}{x+2}\Big).$$

The integral (7) is

$$\frac{1}{4}\int \Big(\frac{1}{x-2} - \frac{1}{x+2}\Big)dx = t + C$$

$$\frac{1}{4}(\ln|x-2| - \ln|x+2|) = t + C$$

$$(\ln|x-2| - \ln|x+2|) = 4t + 4C$$

Exponentiate:

$$\frac{|x-2|}{|x+2|} = e^{4C}e^{4t}$$

$$\frac{x-2}{x+2} = \pm e^{4C}e^{4t}$$

Let K be the constant $\pm e^{4C}$.

$$\frac{x-2}{x+2} = Ke^{4t}.$$

Maybe this is a good time to find K. Plug $t = 0$ into the most recent equation to learn

$$\frac{x_0-2}{x_0+2} = K. \tag{8}$$

(We will keep writing K for now. Eventually, we will insert this value for K into our answer.) We want to solve for x. Multiply both sides by $x+2$:

$$x - 2 = Ke^{4t}(x+2).$$

Subtract $Ke^{4t}x$ from both sides and add 2 to both sides:

$$x - Ke^{4t}x = 2Ke^{4t} + 2.$$

Factor an x out pof both terms on the left:

$$x(1 - Ke^{4t}) = 2Ke^{4t} + 2.$$

Divide both sides by $1 - Ke^{4t}$

$$x = \frac{2Ke^{4t} + 2}{1 - Ke^{4t}}$$

Insert (8) into our answer:

$$x = \frac{2\left(\frac{x_0-2}{x_0+2}\right)e^{4t} + 2}{1 - \left(\frac{x_0-2}{x_0+2}\right)e^{4t}}.$$

Factor a 2 out of both terms in the numerator. Multiply the top and the bottom both by $x_0 + 2$:

$$\boxed{x = \frac{2((x_0-2)e^{4t} + x_0 + 2)}{x_0 + 2 - (x_0-2)e^{4t}}}.$$

Check. We plug the proposed answer into $\frac{dx}{dt} - x^2 + 4$. We some the expression ultimately simplifies to become 0. At any rate, if $x = \frac{2((x_0-2)e^{4t}+x_0+2)}{x_0+2-(x_0-2)e^{4t}}$, then

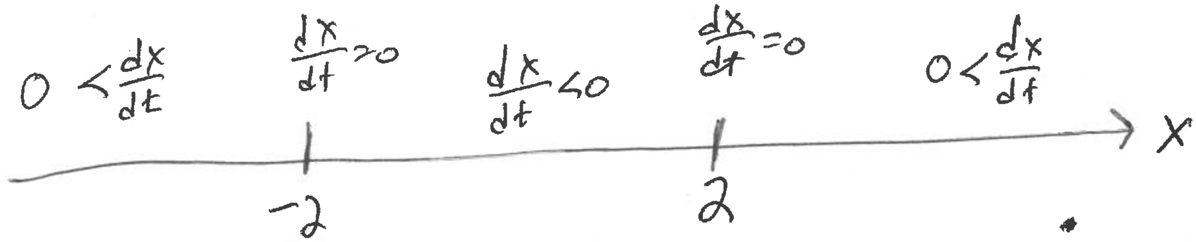
$$\begin{aligned}
&\frac{dx}{dt} - x^2 + 4 \\
&= \begin{cases} \frac{2((x_0-2)e^{4t}+x_0+2)(-1)(-4(x_0-2)e^{4t})}{(x_0+2-(x_0-2)e^{4t})^2} + \frac{8(x_0-2)e^{4t}}{(x_0+2-(x_0-2)e^{4t})}) \\ -\left(\frac{2((x_0-2)e^{4t}+x_0+2)}{x_0+2-(x_0-2)e^{4t}}\right)^2 \\ +4 \end{cases} \\
&= \frac{\begin{cases} +2\Big((x_0-2)e^{4t} + x_0 + 2\Big)(-1)(-4(x_0-2)e^{4t}) \\ +(8(x_0-2)e^{4t})(x_0+2-(x_0-2)e^{4t}) \\ -\Big(2((x_0-2)e^{4t} + x_0 + 2)\Big)^2 \\ +4(x_0+2-(x_0-2)e^{4t})^2 \end{cases}}{(x_0+2-(x_0-2)e^{4t})^2} \\
&= \frac{\begin{cases} +8(x_0-2)^2e^{8t} + 8(x_0+2)(x_0-2)e^{4t} \\ +8(x_0+2)(x_0-2)e^{4t} - 8(x_0-2)^2e^{8t} \\ -4(x_0-2)^2e^{8t} - 8(x_0-2)(x_0+2)e^{4t} - 4(x_0-2)^2 \\ +4(x_0+2)^2 - 8(x_0+2)(x_0-2)e^{4t} + 4(x_0-2)^2e^{8t} \end{cases}}{(x_0+2-(x_0-2)e^{4t})^2}
\end{aligned}$$

The numerator adds to zero. Our proposed solution is correct.

The pictures are on the next page.

Pictures for Problem 5 in Section 2.2

Phase Diagram for $\frac{dx}{dt} = x^2 - 4$



The solution of $\frac{dx}{dt} = x^2 - 4$, $x(0) = x_0$

for a few choices of x_0