Problem 19 in Section 2.1 Consider an alligator population which satisfies the extinction/explosion Differential Equation as in Problem 18. If the initial population is 100 alligators and there are 10 births per month and 9 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10 times the threshold population M?

Solution We saw in number 18 that $M = \frac{D(0)P(0)}{B(0)}$. This problem has

$$P(0) = 100, \quad B(0) = 10, \quad \text{and} \quad D(0) = 9.$$

Thus $M = \frac{9(100)}{10} = 90$. We want to find t with $P(t) = (90)(10) = 900$. The solution of the Initial Value Problem

$$\frac{dP}{dt} = kP(P - M) \quad P(0) = P_0 \quad \text{with } k \text{ and } M \text{ positive}$$

is

$$P = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}.$$

(See problem 33 or the class notes. There is no reason to memorize this formula.) Of course the Differential Equations

$$\frac{dP}{dt} = kP^2 - kMP$$

$$\frac{dP}{dt} = aP^2 - bP$$

are exactly the same if one takes $a = k$ and $b = kM$, where $B = aP^2$ and $D = bP$. In particular, $k = a = \frac{B(0)}{P(0)^2} = \frac{B_0}{P_0^2} = \frac{10}{100^2} = \frac{1}{1000}$. Thus,

$$P = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}} = \frac{90(100)}{100 + (-10)e^{\frac{90}{1000}t}}.$$

Our job is to find t with

$$900 = \frac{90(100)}{100 + (-10)e^{\frac{90}{1000}t}}.$$

Multiply both sides by $100 - 10e^{9t/100}$. Divide both sides by 900.

$$100 - 10e^{\frac{9}{100}t} = 10.$$

Subtract 10 from both sides. Add $10e^{\frac{9}{100}t}$ to both sides. Obtain

$$90 = 10e^{\frac{9}{100}t}.$$

$$9 = e^{\frac{9}{100}t}$$

$$\ln 9 = \frac{9}{100}t$$

$$\frac{100}{9}\ln 9 = t.$$

$$\boxed{\text{The population will reach 900, which is } 10M\text{, after } \frac{100}{9}\ln 9 \text{ months.}}$$