# Math 241, Exam 1, Spring, 2023

**You should KEEP this piece of paper.** Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

**No Calculators, Cell phones, computers, notes, etc.**

(1) **Find a system of parametric equations for the line through the points $P_1 = (2,4,5)$ and $P_2 = (3,4,7)$. Check your answer. Make sure it is correct.**

Observe that $\overrightarrow{P_1P_2} = \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{k}}$. The answer is

$$\boxed{x = 2 + t, \quad y = 4, \quad z = 5 + 2t}.$$

**Check:** At $t = 0$, the line hits $(2,4,5)$. At $t = 1$, the line hits $(3,4,7)$.

(2) **Find an equation for the plane through the points $P_1 = (1,-1,2)$, $P_2 = (2,4,-1)$, and $P_3 = (3,2,1)$. Check your answer. Make sure it is correct.**

Observe that

$$\overrightarrow{P_1P_2} = \vec{\boldsymbol{i}} + 5\vec{\boldsymbol{j}} - 3\vec{\boldsymbol{k}} \text{ and } \overrightarrow{P_1P_3} = 2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}} - 1\vec{\boldsymbol{k}}.$$

It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 1 & 5 & -3 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} \vec{\boldsymbol{k}}$$

$$= 4\vec{\boldsymbol{i}} - 5\vec{\boldsymbol{j}} - 7\vec{\boldsymbol{k}}.$$

The plane through $(1,-1,2)$ perpendicular to $-4\vec{\boldsymbol{i}} - 5\vec{\boldsymbol{j}} - 7\vec{\boldsymbol{k}}$ is

$$4(x-1) - 5(y+1) - 7(z-2) = 0$$

or

$$\boxed{4x - 5y - 7z = -5}.$$

**Check.** Plug $(1,-1,2)$ into the proposed answer:

$$4(1)-5(-1)-7(2)=-5\checkmark$$

Plug $(2,4,-1)$ into the proposed answer:

$$4(2)-5(4)-7(-1)=-5\checkmark$$

Plug $(3,2,1)$ into the proposed answer:

$$4(3)-5(2)-7(1)=-5\checkmark$$

(3) **Express $\vec{v}=2\vec{i}+5\vec{j}$ as the sum of a vector parallel to $\vec{w}=-\vec{i}+4\vec{j}$ and a vector orthogonal to $\vec{w}$. Check your answer. Make sure it is correct.**

We compute

$$\operatorname{proj}_{\vec{w}}\vec{v}=\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}=\frac{-2+20}{17}\vec{w}=\frac{18}{17}(-\vec{i}+4j).$$

Thus,

$$\vec{v}-\operatorname{proj}_{\vec{w}}\vec{v}=(2\vec{i}+5\vec{j})-\frac{18}{17}(-\vec{i}+4\vec{j})=\frac{13}{17}(4\vec{i}+\vec{j}).$$

We conclude that

$\vec{v}=\frac{18}{17}(-\vec{i}+4j)+\frac{13}{17}(4\vec{i}+\vec{j})$ with $\frac{18}{17}(-\vec{i}+4j)$ parallel to $\vec{w}$ and $\frac{13}{17}(4\vec{i}+\vec{j})$ perpendicular to $\vec{w}$.

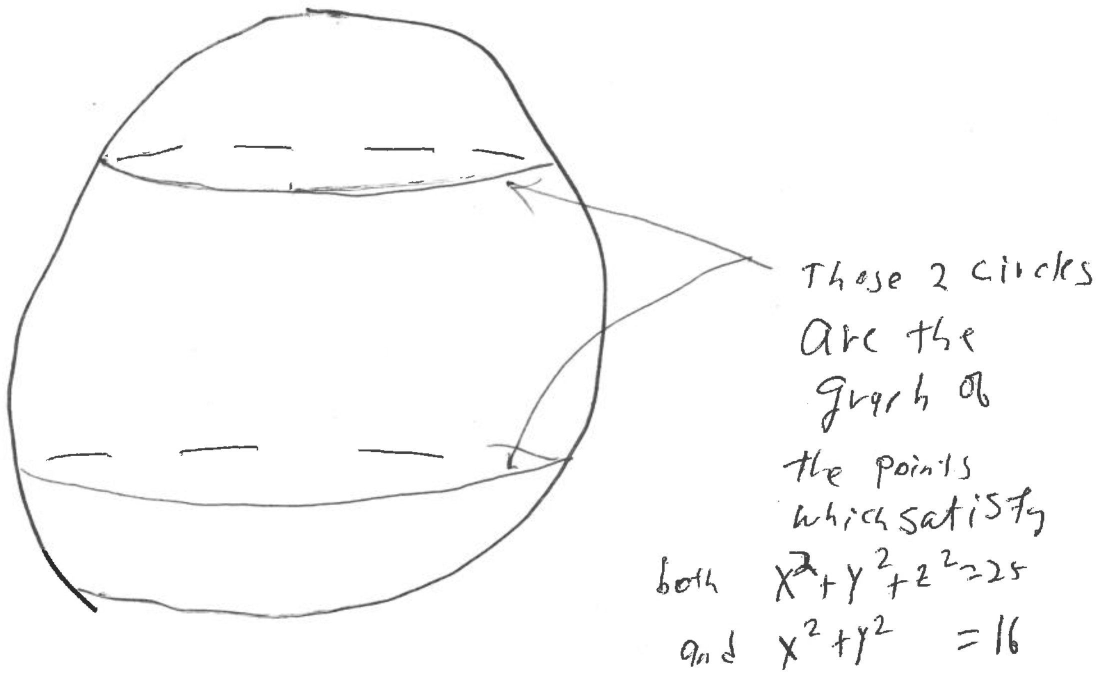
**Check:** It is clear that $\vec{v}=\frac{18}{17}(-\vec{i}+4j)+\frac{13}{17}(4\vec{i}+\vec{j})$, $\frac{18}{17}(-\vec{i}+4j)$ is parallel to $\vec{w}$ and $\frac{13}{17}(4\vec{i}+\vec{j})$ is perpendicular to $\vec{w}$.

(4) **Name, describe, and graph the set of all points in three-space which satisfy both equations $x^2+y^2+z^2=25$ and $x^2+y^2=16$.**

The points which satisfy $x^2+y^2+z^2=25$ form the sphere with radius $5$ and center the origin. The points which satisfy $x^2+y^2=16$ form the cylinder with radius $4$ and the $z$-axis in its center. The set of points which satisfy both $x^2+y^2+z^2=25$ and $x^2+y^2=16$ form two circles. Both circles are in planes parallel to the $xy$-plane and both circles have radius $3$. One of the circles has center $(0,0,3)$; the other circle has center $(0,0,-3)$.

The picture is on the next page.

The picture for 4.



(5) **Find the point on the line**

$$x = 6 + 2t, \quad y = 7 + 3t, \quad z = 8 + 4t$$

**which is closest to the point** $(1, 2, 3)$**.**

The plane though $(1, 2, 3)$ perpedicular to the line

$$x = 6 + 2t, \quad y = 7 + 3t, \quad z = 8 + 4t$$

is

$$2(x - 1) + 3(y - 2) + 4(z - 3) = 0$$

or

$$2x + 3y + 4z = 20.$$

The answer is the intersection of the line and the plane. First we find WHEN the intersection occurs:

$$2(6 + 2t) + 3(7 + 3t) + 4(8 + 4t) = 20;$$

$$4t + 9t + 16t = 20 - 12 - 21 - 32;$$

$$29t = -45;$$

$$t = \frac{-45}{29}.$$

The intersection occurs at

$$x = 6 - \tfrac{90}{29} = \tfrac{84}{29};$$

$$y = 7 - \tfrac{135}{29} = \tfrac{68}{29}; \text{ and}$$

$$z = 8 - \tfrac{180}{29} = \tfrac{52}{29}.$$

$$\boxed{(\tfrac{84}{29}, \tfrac{68}{29}, \tfrac{52}{29})}.$$

**CHECK!** The proposed answer is on the line (when $t = \frac{-45}{29}$) and the vector from $(\frac{84}{29}, \frac{68}{29}, \frac{52}{29})$ to $(1, 2, 3)$ is $\frac{-55}{29}\overrightarrow{\boldsymbol{i}} - \frac{10}{29}\overrightarrow{\boldsymbol{j}} + \frac{35}{29}\overrightarrow{\boldsymbol{k}}$. This vector is perpendicular to the line because

$$(\tfrac{-55}{29}\overrightarrow{\boldsymbol{i}} - \tfrac{10}{29}\overrightarrow{\boldsymbol{j}} + \tfrac{35}{29}\overrightarrow{\boldsymbol{k}}) \cdot (2\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}} + 4\overrightarrow{\boldsymbol{k}}) = 0.$$