Quiz for March 28, 2008

Find the absolute extreme points of $f(x,y,z) = 2x + y - 2z$ subject to the constraint $x^2 + y^2 + z^2 = 4$.

Answer: Let $g(x,y,z) = x^2 + y^2 + z^2$. We see that $\overrightarrow{\nabla} g = 2x\overrightarrow{\boldsymbol{i}} + 2y\overrightarrow{\boldsymbol{j}} + 2z\overrightarrow{\boldsymbol{k}}$. The only point where $\overrightarrow{\nabla} g = 0$ is the origin and the origin is not on the constraint $g = 4$. There are no endpoints. So, the absolute extreme points of f on $g = 4$ occur at points where $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$ for some constant λ.

We find all points on $g = 4$ with $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$ for some constant λ.

We find all points on $g = 4$ with $2\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} - 2\overrightarrow{\boldsymbol{k}} = \lambda(2x\overrightarrow{\boldsymbol{i}} + 2y\overrightarrow{\boldsymbol{j}} + 2z\overrightarrow{\boldsymbol{k}})$ for some constant λ.

We find all points (x,y,z) and all numbers λ with

$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ 2 = 2x\lambda \\ 1 = 2y\lambda \\ -2 = 2z\lambda. \end{cases}$$

We see that λ can not be zero (because $2 \neq 0$) so we find all points (x,y,z) and all numbers λ with

$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ \frac{1}{\lambda} = x \\ \frac{1}{2\lambda} = y \\ \frac{-1}{\lambda} = z. \end{cases}$$

We find all points (x,y,z) and all numbers λ with

$$\begin{cases} (\frac{1}{\lambda})^2 + (\frac{1}{2\lambda})^2 + (\frac{-1}{\lambda})^2 = 4 \\ \frac{1}{\lambda} = x \\ \frac{1}{2\lambda} = y \\ \frac{-1}{\lambda} = z. \end{cases}$$

We find all points (x,y,z) and all numbers λ with

$$\begin{cases} (\frac{1}{\lambda})^2(1 + 1/4 + 1) = 4 \\ \frac{1}{\lambda} = x \\ \frac{1}{2\lambda} = y \\ \frac{-1}{\lambda} = z. \end{cases}$$

We find all points (x,y,z) and all numbers λ with

$$\begin{cases} \frac{9}{16} = \lambda^2 \\ \frac{1}{\lambda} = x \\ \frac{1}{2\lambda} = y \\ \frac{-1}{\lambda} = z. \end{cases}$$

If $\lambda = 3/4$, then $(x, y, z) = (4/3, 2/3, -4/3)$ and $f(x, y, z) = 6$
If $\lambda = -3/4$, then $(x, y, z) = (-4/3, -2/3, 4/3)$ and $f(x, y, z) = -6$.

We conclude that the maximum value of f on $g = 4$ is 6
and $f(4/3, 2/3, -4/3) = 6$.
Also, the minimum value of f on $g = 4$ is -6
and this value occurs at $f(-4/3, -2/3, 4/3) = -6$.