Please PRINT your name ______________________________________

No calculators, cell phones, computers, notes, etc.

Circle your answer. $\boxed{\text{Make your work correct, complete and coherent.}}$

Please take a picture of your quiz (for your records) just before you turn the quiz in. I will e-mail your grade and my comments to you. I will keep your quiz.

The quiz is worth 5 points. The solutions will be posted on my website later today.

Quiz 6, March 29, 2023

Find the points on the ellipse $x^2+2y^2=1$ where $f(x,y)=xy$ has its extreme values.

Answer:

This is a Lagrange multiplier problem. We want to optimize $f(x,y)=xy$ subject to the constraint $g(x,y)=x^2+2y^2$ is equal to 1. We want to find all triples (x,y,λ) such that

$$\begin{cases} g(x,y)=1 \text{ and} \\ \overrightarrow{\nabla} f|_{(x,y)}=\lambda\overrightarrow{\nabla} g|_{(x,y)}. \end{cases}$$

We want

$$\begin{cases} x^2+2y^2=1 \text{ and} \\ y\overrightarrow{\boldsymbol{i}}+x\overrightarrow{\boldsymbol{j}}=\lambda(2x\overrightarrow{\boldsymbol{i}}+4y\overrightarrow{\boldsymbol{j}}). \end{cases}$$

We want

$$\begin{cases} x^2+2y^2=1, \\ y=2x\lambda, \text{ and} \\ x=4y\lambda. \end{cases}$$

Substitute $2x\lambda$ for y. We want

$$\begin{cases} x^2+2y^2=1, \\ y=2x\lambda, \text{ and} \\ x=4(2x\lambda)\lambda. \end{cases}$$

The bottom equation says that either $x=0$ or $x\neq 0$ and $\frac{1}{8}=\lambda^2$. If $x=0$, then the top equation gives $y=\pm\frac{1}{\sqrt{2}}$. So, $(0,\frac{1}{\sqrt{2}})$ and $(0,-\frac{1}{\sqrt{2}})$ are points of interest. If $\frac{1}{8}=\lambda^2$, then $\lambda=\pm\frac{1}{2\sqrt{2}}$. If $\lambda=\frac{1}{2\sqrt{2}}$, then $y=2x\lambda$ becomes $y=\frac{2x}{2\sqrt{2}}$ and $x^2+2y^2=1$ becomes $x^2+2\frac{x^2}{2}=1$. In this last case, $2x^2=1$, or $x=\pm\frac{1}{\sqrt{2}}$. Plug $x=\pm\frac{1}{\sqrt{2}}$ into the equation for the ellipse. We pick up four more points of interest; namely,

$$(\tfrac{1}{\sqrt{2}},\tfrac{1}{2}),\ (\tfrac{1}{\sqrt{2}},-\tfrac{1}{2}),\ (-\tfrac{1}{\sqrt{2}},\tfrac{1}{2}),\ (-\tfrac{1}{\sqrt{2}},-\tfrac{1}{2}).$$

The extreme values of f subject to the constraint $g = 1$ occur at one of the six points of interest. We plug these points into f

$$\begin{aligned}
f(0, \tfrac{1}{\sqrt{2}}) &= 0 \\
f(0, -\tfrac{1}{\sqrt{2}}) &= 0 \\
f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}) &= \tfrac{1}{2\sqrt{2}} \\
f(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}) &= -\tfrac{1}{2\sqrt{2}} \\
f(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}) &= -\tfrac{1}{2\sqrt{2}} \\
f(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}) &= \tfrac{1}{2\sqrt{2}}
\end{aligned}$$

The maximum of f subject to $g = 1$ occurs at $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2\sqrt{2}})$.
The minimum of f subject to $g = 1$ occurs at $(-\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2\sqrt{2}})$