Math 241, Exam 3, Spring, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) (8 points) **Find the point on the line**

$$x = t, \quad y = 2t + 1, \quad z = 4t + 3$$

which is closest to the point $(1, 6, 16)$**. DEMONSTRATE that your answer is correct.**

The vector $\vec{\boldsymbol{v}} = \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 4\vec{\boldsymbol{k}}$ is parallel to the given line. The plane $(x - 1) + 2(y - 6) + 4(z - 16) = 0$ is perpendicular to the given line and passes through the given point. This plane may also be written as $x+2y+4z = 77$. The answer is the intersection of the line and the plane. The line and the plane intersect when

$$t + 2(2t + 1) + 4(4t + 3) = 77.$$

Thus $21t = 63$ and $t = 3$. The point of intersection is $\boxed{(3, 7, 15)}$.

Check. The point $(3, 7, 15)$ is on the line (when $t = 3$) and the vector which from $(1, 6, 16)$ to $(3, 7, 15)$ is $2\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}$, which IS perpendicular to the line.

(2) (8 points) **Find the maximum and minimum of** $f = 3x+4y$ **on** $x^2+y^2 = 1$**.**

This is a Lagrange multiplier problem. We look for all points (x, y) and all scalars λ so that

$$x^2 + y^2 = 1 \quad \text{and} \quad \vec{\nabla} f = \lambda \vec{\nabla}(x^2 + y^2),$$

simultaneously. We want to solve

$$x^2 + y^2 = 1 \quad \text{and} \quad 3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} = \lambda(2x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}}),$$

simultaneously. We want to solve

$$x^2 + y^2 = 1 \quad \text{and} \quad 3 = 2x\lambda \quad \text{and} \quad 4 = 2y\lambda,$$

simultaneously. We see that λ can not be zero (because $3 = 0$ makes no sense!); consequently, it does no harm to divide by λ. We want to solve

$$x^2 + y^2 = 1 \quad \text{and} \quad \frac{3}{2\lambda} = x \quad \text{and} \quad \frac{4}{2\lambda} = y,$$

simultaneously. We want to solve

$$(\tfrac{3}{2\lambda})^2 + (\tfrac{4}{2\lambda})^2 = 1 \quad \text{and} \quad \frac{3}{2\lambda} = x \quad \text{and} \quad \frac{4}{2\lambda} = y,$$

simultaneously. The first equation says $9+16 = 4\lambda^2$; so $\lambda = \pm\frac{5}{2}$. If $\lambda = \frac{5}{2}$, then $x = \frac{3}{5}$ and $y = \frac{4}{5}$. If $\lambda = -\frac{5}{2}$, then $x = -\frac{3}{5}$ and $y = -\frac{4}{5}$. There are two points of interest. We plug these points into f:

$$f(\tfrac{3}{5}, \tfrac{4}{5}) = 5 \quad \text{and} \quad f(-\tfrac{3}{5}, -\tfrac{4}{5}) = -5.$$

We conclude that the maximum of f on $x^2 + y^2 = 1$ occurs at $(\frac{3}{5}, \frac{4}{5}, 5)$ and the minimum of f on $x^2 + y^2 = 1$ occurs at $(-\frac{3}{5}, -\frac{4}{5}, -5)$.

(3) (8 points) **Find the absolute extreme points of**

$$f(x, y) = x^2 + xy + y^2 - 3x + 3y$$

on the triangular region cut from the first quadrant by the line $x + y = 4$**.**

There is a picture of the domain on the last page.

We look for interior points where both partial derivatives vanish. We calculate $f_x = 2x + y - 3$ and $f_y = x + 2y + 3$. Both partial derivatives are zero when $3 - 2x = y$ and $0 = x + 2(3 - 2x) + 3$. The second equation is $0 = x + 6 - 4x + 3$ or $0 = 9 - 3x$, hence $x = 3$. The first equation gives $y = 3 - 6 = -3$. The point $(3, -3)$ is not in our domain.

The three vertices $(0, 0)$, $(4, 0)$, and $(0, 4)$ are points of interest.

Now we look at f restricted to each of the three parts of the boundary.

The vertical part of the boundary is $y = 0$. We see that f restricted to $y = 0$ is $f|_{y=0} = x^2 - 3x$. We compute $\frac{d}{dx}(f|_{y=0}) = 2x - 3$. This derivative is zero when $x = \frac{3}{2}$. Thus, $(\frac{3}{2}, 0)$ is a point of interest.

The horizontal part of the boundary is $x = 0$. We see that f restricted to $x = 0$ is $f|_{x=0} = y^2 + 3y$. We compute $\frac{d}{dy}(f|_{x=0}) = 2y + 3$. This derivative is never zero in our domain!

The slanty part of the boundary is $x + y = 4$. The restriction of f to this part of the boundary is

$$f|_{y=4-x} = x^2 + x(4 - x) + (4 - x)^2 - 3x + 3(4 - x) = x^2 - 10x + 28.$$

The derivative $\frac{d}{dx}(x^2 - 10x + 28) = 2x - 10$. This derivative is zero when $x = 5$ and $y = -1$. The point $(5, -1)$ is not in our domain.

We plug all points of interest into f to see that

$$\begin{aligned} f(0,0) &= 0 \\ f(4,0) &= 16 - 12 = 4 \\ f(0,4) &= 16 + 12 = 28 \\ f(\tfrac{3}{2}, 0) &= \tfrac{9}{4} - \tfrac{9}{2} = -\tfrac{9}{4} \end{aligned}$$

$$\boxed{\begin{array}{l}\text{The maximum of } f \text{ on the domain occurs at } (0,4,28).\\ \text{The minimum of } f \text{ on the domain occurs at } (\tfrac{3}{2}, 0, -\tfrac{9}{4}).\end{array}}$$

(4) (8 points) **Find the local maxima, local minima, and saddle points of $f(x, y) = 2x^3 + 3xy + 2y^3$.**

We compute $f_x = 6x^2 + 3y$ and $f_y = 3x + 6y^2$. Both partial derivatives are zero when $y = -2x^2$ and $0 = 3x + 6(-2x^2)^2$. The second equation is $0 = 3x + 24x^4$ or $0 = 3x(1 + 8x^3)$. Thus, $x = 0$ or $x = -\frac{1}{2}$. When $x = 0$, the first equation gives $y = 0$. When $x = -\frac{1}{2}$, the first equation gives $y = -\frac{1}{2}$. We apply the second derivative test at each critical point. We compute

$$f_{xx} = 12x, \quad f_{xy} = 3, \quad f_{yy} = 12y.$$

So

$$H = f_{xx}f_{yy} - f_{xy}^2 = 144xy - 9.$$

We see that

$$H(0,0) = -9 < 0, \quad H(-\frac{1}{2}, -\frac{1}{2}) = 48 - 9 > 0, \quad f_{xx}(-\frac{1}{2}, -\frac{1}{2}) = -6 < 0.$$

Thus,

$$\boxed{\begin{array}{l}(0, 0, f(0,0)) \text{ is a saddle point}\\ \text{and } (-\tfrac{1}{2}, -\tfrac{1}{2}, f(-\tfrac{1}{2}, -\tfrac{1}{2})) \text{ is a local maximum.}\end{array}}$$

(5) (9 points) **Find the area of the region bounded by $x = -y^2$ and $y = x + 2$. (You must draw a meaningful picture.)**

The curves $x = -y^2$ and $y = x + 2$ intersect when $y = -y^2 + 2$ so $y^2 + y - 2 = 0$, or $(y + 2)(y - 1) = 0$. Thus $y = -2$ or $y = 1$. The points of intersection are $(-4, -2)$ and $(-1, 1)$. We drew the picture on the last page. We better fill up the region using horizontal lines. The area is equal to

$$\int_{-2}^{1}\int_{y-2}^{-y^2} dx\, dy = \int_{-2}^{1}(-y^2 - y + 2)dy = -\tfrac{y^3}{3} - \tfrac{y^2}{2} + 2y|_{-2}^{1}$$

$$= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - (\tfrac{8}{3} - 2 - 4) = \boxed{\tfrac{9}{2}}.$$

(6) (9 points) **Find the length of the curve whose position vector at time** t **is**

$$\overrightarrow{\boldsymbol{r}}(t) = (2\cos t)\overrightarrow{\boldsymbol{i}} + (2\sin t)\overrightarrow{\boldsymbol{j}} + 2t\overrightarrow{\boldsymbol{k}},$$

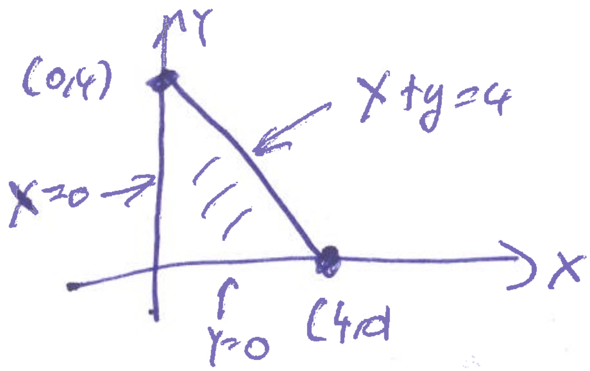
for $0 \leq t \leq \frac{\pi}{4}$**.**

Arc length is equal to

$$\int_0^{\pi/4} |\overrightarrow{\boldsymbol{r}}'(t)| dt = \int_0^{\pi/4} |-(2\sin t)\overrightarrow{\boldsymbol{i}} + (2\cos t)\overrightarrow{\boldsymbol{j}} + 2\overrightarrow{\boldsymbol{k}}| dt$$

$$= \int_0^{\pi/4} \sqrt{4\sin^2 t + 4\cos^2 t + 4} dt = \boxed{\sqrt{8}\frac{\pi}{4}}$$

The domain for Problem 3



The picture for number 5