Math 241, Exam 2, Spring, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Problem 2 is worth 8 points; each of the other problems is worth 7 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the point on the line**

$$x = -t + 2, \quad y = t + 1, \quad z = 2t - 1$$

which is closest to the point $(2, 3, 1)$**. DEMONSTRATE that your answer is correct.**

Let $\mathcal{P}$ be the plane through $(2, 3, 1)$ which is perpendicular to the line

$$\ell : \quad x = -t + 2, \quad y = t + 1, \quad z = 2t - 1.$$

The answer is the intersection of ℓ and $\mathcal{P}$. Of course, the answer is on ℓ and the vector which connects the answer to $(2, 3, 1)$ is perpendicular to ℓ.

The plane $\mathcal{P}$ passes through $(2, 3, 1)$ and is perpendicular to

$$\overrightarrow{\boldsymbol{v}} = -\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + 2\overrightarrow{\boldsymbol{k}}.$$

This plane is $-(x - 2) + (y - 3) + 2(z - 1) = 0$. In other words, $\mathcal{P}$ is $-x + y + 2z = 3$.

The plane $\mathcal{P}$ and the line ℓ intersect when

$$-(-t + 2) + (t + 1) + 2(2t - 1) = 3 \quad \text{or} \quad t + t + 4t = 3 + 2 - 1 + 2.$$

Thus, $6t = 6$ and $t = 1$. When $t = 1$, the line touches $\boxed{(1, 2, 1).}$

Check. The proposed answer is on ℓ and the vector which goes from the proposed answer to $(2, 3, 1)$ is $\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}}$, which is perpendicular to ℓ.

(2) **Find an equation for the plane through the points $P_1 = (2, -1, 3)$, $P_2 = (2, 4, 1)$, and $P_3 = (1, 2, 3)$. DEMONSTRATE that your answer is correct.**

Observe that $\overrightarrow{P_1P_2} = 5\vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}}$ and $\overrightarrow{P_1P_3} = -\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}$. It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 0 & 5 & -2 \\ -1 & 3 & 0 \end{vmatrix} = \vec{\boldsymbol{i}} \begin{vmatrix} 5 & -2 \\ -3 & 0 \end{vmatrix} - \vec{\boldsymbol{j}} \begin{vmatrix} 0 & -2 \\ -1 & 0 \end{vmatrix} + \vec{\boldsymbol{k}} \begin{vmatrix} 0 & 5 \\ -1 & 3 \end{vmatrix}$$

$$= 6\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 5\vec{\boldsymbol{k}}.$$

The plane through $(2, -1, 3)$ perpendicular to $6\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 5\vec{\boldsymbol{k}}$ is

$$6(x-2) + 2(y+1) + 5(z-3) = 0.$$

We rewrite this equation as

$$\boxed{6x + 2y + 5z = 25}$$

Check.

The point $(2, -1, 3)$ satisfies our answer:

$$6(2) + 2(-1) + 5(3) = 25.$$

The point $(2, 4, 1)$ satisfies our answer:

$$6(2) + 2(4) + 5(1) = 25.$$

The point $(1, 2, 3)$ satisfies our answer:

$$6(1) + 2(2) + 5(3) = 25.$$

(3) **Express $\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}$ as the sum of a vector parallel to $\vec{\boldsymbol{w}} = -\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}$ and a vector orthogonal to $\vec{\boldsymbol{w}}$. DEMONSTRATE that your answer is correct.**

$$\text{proj}_{\vec{\boldsymbol{w}}} \vec{\boldsymbol{v}} = \tfrac{\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}}{\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}} \vec{\boldsymbol{w}} = \tfrac{-2+12}{1+16} \vec{\boldsymbol{w}}$$
$$= \tfrac{10}{17}(-\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}).$$

$$\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{w}}} \vec{\boldsymbol{v}} = (\tfrac{34}{17}\vec{\boldsymbol{i}} + \tfrac{51}{17}\vec{\boldsymbol{j}}) - (-\tfrac{10}{17}\vec{\boldsymbol{i}} + \tfrac{40}{17}\vec{\boldsymbol{j}}) = \tfrac{44}{17}\vec{\boldsymbol{i}} + \tfrac{11}{17}\vec{\boldsymbol{j}}$$
$$= \tfrac{11}{17}(4\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}}).$$

$$\boxed{\begin{array}{l}\vec{v} = \frac{10}{17}(-\vec{i} + 4\vec{j}) + \frac{11}{17}(4\vec{i} + \vec{j}),\\ \text{with } \frac{10}{17}(-\vec{i} + 4\vec{j}) \text{ parallel to } \vec{w} \text{ and}\\ \frac{11}{17}(4\vec{i} + \vec{j})) \text{ perpendicular to } \vec{w}.\end{array}}$$

Check. We see that

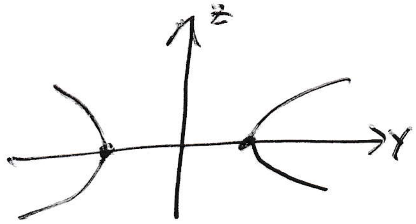
$$\frac{10}{17}(-\vec{i} + 4\vec{j}) + \frac{11}{17}(4\vec{i} + \vec{j}) = \frac{34}{17}\vec{i} + \frac{51}{17}\vec{j} = 2\vec{i} + 3\vec{j} = \vec{v}.$$

We see that $\frac{10}{17}(-\vec{i} + 4\vec{j})$ is parallel to $\vec{v} = (-\vec{i} + 4\vec{j})$.
We see that $\frac{11}{17}(4\vec{i} + \vec{j}) \cdot \vec{w} = \frac{11}{17}(4\vec{i} + \vec{j}) \cdot (-\vec{i} + 4\vec{j}) = \frac{11}{17}(4-4) = 0$.

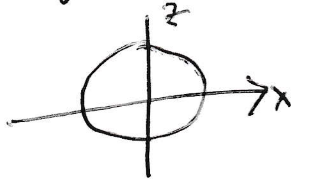
(4) **Name, describe, and graph the set of all points in three-space which satisfy the equation $y^2 - x^2 - z^2 = 1$. Is the graph a curve, a surface, or a solid?**

$y^2 - x^2 - z^2 = 1$

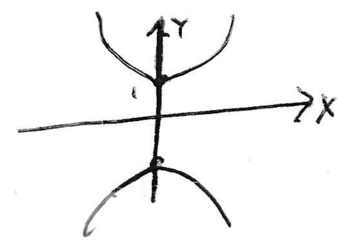
When $x=0$, the equation is $y^2 - z^2 = 1$. The graph of this equation is a hyperbola



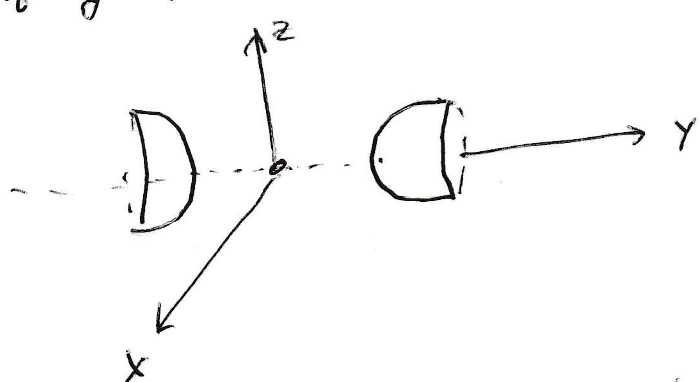
When $y=0$, there is no graph because $-x^2 - z^2 = 1$ has no solutions
when $y = +1$ or -1, then the graph is just one point
When $y > 1$ or $y < -1$, then the graph is a circle Pos # $= x^2 + z^2$



When $z=0$, the graph is a hyperbola
$y^2 - x^2 = 1$



The graph of $y^2 - x^2 - z^2 = 1$ is a hyperboloid of two sheets



The graph is a surface. Take the hyperbola $y^2 - z^2 = 1$ in the yz-plane and rotate it about the y-axis

(5) **The position vector of an object at time t is**

$$\overrightarrow{\boldsymbol{r}}(t) = \cos 2t\,\overrightarrow{\boldsymbol{i}} + \sin 2t\,\overrightarrow{\boldsymbol{j}} + t\,\overrightarrow{\boldsymbol{k}}.$$

How far does the object travel between $t = 0$ and $t = \pi$?

$$\int_0^\pi |\overrightarrow{\boldsymbol{r}}'(t)|dt = \int_0^\pi |(-2\sin 2t)\,\overrightarrow{\boldsymbol{i}} + (2\cos 2t)\,\overrightarrow{\boldsymbol{j}} + \overrightarrow{\boldsymbol{k}}|dt = \int_0^\pi \sqrt{5}dt$$

$$= \boxed{\sqrt{5}\pi}.$$

(6) **If $z = 2x^2y^4e^{3x^3y^2} + x^2y^3\sin(xy)$, then find $\frac{\partial z}{\partial x}$.**

$$\boxed{\frac{\partial z}{\partial x} = 2x^2y^4e^{3x^3y^2}9x^2y^2 + 4xy^4e^{3x^3y^2} + x^2y^4\cos(xy) + 2xy^3\sin(xy).}$$

(7) **Find the directional derivative of $f(x, y) = x^2y^3$ in the direction $\overrightarrow{\boldsymbol{v}} = 2\,\overrightarrow{\boldsymbol{i}} + 3\,\overrightarrow{\boldsymbol{j}}$ at the point $P = (2, -2)$.**

$$(D_{\overrightarrow{\boldsymbol{v}}}f)|_P = (\overrightarrow{\nabla}f)|_P \cdot \frac{\overrightarrow{\boldsymbol{v}}}{|\overrightarrow{\boldsymbol{v}}|} = (2xy^3\,\overrightarrow{\boldsymbol{i}} + 3x^2y^2\,\overrightarrow{\boldsymbol{j}})|_{(2,-2)} \cdot \frac{2\,\overrightarrow{\boldsymbol{i}}+3\,\overrightarrow{\boldsymbol{j}}}{\sqrt{13}}$$

$$= (-32\,\overrightarrow{\boldsymbol{i}} + 48\,\overrightarrow{\boldsymbol{j}}) \cdot \frac{2\,\overrightarrow{\boldsymbol{i}}+3\,\overrightarrow{\boldsymbol{j}}}{\sqrt{13}} = \boxed{\frac{-64 + 144}{\sqrt{13}}}.$$