Math 174, Fall 2003, Solution to Quiz 4

Question: Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all integers $n \geq 1$.

Answer: Proof by induction. For each integer n with $n \geq 1$, let $P(n)$ be the statement:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Base case. We show that $P(1)$ is true. Well, $P(1)$ is the statement

$$\frac{1}{1\cdot 2}=\frac{1}{1+1}.$$

This is indeed true.

Inductive step. We assume that $P(n)$ is true for some n. We must show that $P(n+1)$ is true. In other words, we assume that

(IH)
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

We must prove that

(*)
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}.$$

The left side of (*) is

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right]+\frac{1}{(n+1)(n+2)},$$

which according to (IH) is equal to

$$\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}$$

$$=\frac{(n+1)}{(n+2)},$$

and this is the right side of (*). We have completed our proof. We conclude that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all integers $n \geq 1$.