

MATH 142, EXAM 3, SPRING, 2004

PRINT Your Name: _____

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. CIRCLE your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **Find** $\int \sin^3 x \cos^2 x \, dx$. **Check your answer.**

Save one $\sin x$; convert the remaining $\sin^2 x$ to $1 - \cos^2 x$. The original problem is

$$\int (1 - \cos^2 x) \cos^2 x \sin x \, dx.$$

Let $u = \cos x$. It follows that $du = -\sin x \, dx$, and the integral is

$$\begin{aligned} - \int (1 - u^2) u^2 \, du &= - \int (u^2 - u^4) \, du = - \left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C \\ &= \boxed{- \left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} \right) + C}. \end{aligned}$$

Check. The derivative of the proposed answer is

$$- (\cos^2 x (-\sin x) - \cos^4 x (-\sin x)) = \sin x \cos^2 x (1 - \cos^2 x). \checkmark$$

2. **Find** $\int x \ln x \, dx$. **Check your answer.**

Use integration by parts. Let $u = \ln x$ and $dv = x \, dx$. It follows that $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$. The original problem is

$$\int u \, dv = uv - \int v \, du = \frac{x^2 \ln x}{2} - \int \frac{x}{2} \, dx = \boxed{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C}.$$

Check. The derivative of the proposed answer is

$$\frac{x}{2} + x \ln x - \frac{x}{2}. \checkmark$$

3. **Find** $\int \frac{\ln x}{x} \, dx$. **Check your answer.**

Let $u = \ln x$. It follows that $du = \frac{dx}{x}$. The original integral is equal to

$$\int u du = \frac{u^2}{2} + C = \boxed{\frac{(\ln x)^2}{2} + C.}$$

Check. The derivative of the proposed answer is

$$\ln x \cdot \frac{1}{x} \checkmark$$

4. **Find** $\int \frac{4x^2 - 2x + 1}{x(x^2 + 1)} dx$. **Check your answer.**

Write

$$\frac{4x^2 - 2x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiply both sides by $x(x^2 + 1)$ to obtain

$$4x^2 - 2x + 1 = A(x^2 + 1) + (Bx + C)x;$$

which is the same as

$$4x^2 - 2x + 1 = (A + B)x^2 + Cx + A.$$

Equate the corresponding coefficients to see

$$4 = A + B, \quad -2 = C, \quad 1 = A.$$

So, $B = 3$. Check what we have so far:

$$\frac{1}{x} + \frac{3x - 2}{x^2 + 1} = \frac{x^2 + 1 + 3x^2 - 2x}{x(x^2 + 1)} = \frac{4x^2 - 2x + 1}{x(x^2 + 1)} \checkmark$$

So, the original integral is equal to

$$\int \frac{1}{x} + \frac{3x - 2}{x^2 + 1} dx = \boxed{\ln |x| + \frac{3}{2} \ln(x^2 + 1) - 2 \arctan x + C.}$$

5. **Find** $\int \frac{x+1}{(x-1)^2} dx$. **Check your answer.**

Write

$$\frac{x + 1}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2}.$$

Multiply both sides by $(x - 1)^2$ to get

$$x + 1 = A(x - 1) + B;$$

that is,

$$x + 1 = Ax + (B - A).$$

Equate the corresponding coefficients to conclude that $A = 1$ and $B - A = 1$; so, $B = 2$. We check what we have so far:

$$\frac{1}{x-1} + \frac{2}{(x-1)^2} = \frac{x-1+2}{(x-1)^2} = \frac{x+1}{(x-1)^2} \checkmark$$

So the original integral is

$$\int \frac{1}{x-1} + \frac{2}{(x-1)^2} dx = \boxed{\ln|x-1| - \frac{2}{x-1} + C.}$$

6. **Find** $\int \sqrt{1-x^2} dx$. **Check your answer.**

This integral contains an ugly $a^2 - u^2$; so, I let $u = a \sin \theta$. That is, I let $x = \sin \theta$. It follows that $dx = \cos \theta d\theta$. Observe that

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta.$$

The original integral is equal to

$$\begin{aligned} \int \cos^2 \theta d\theta &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C = \frac{1}{2} \left(\arcsin x + \frac{2 \sin \theta \cos \theta}{2} \right) + C \\ &= \boxed{\frac{1}{2} \left(\arcsin x + x \sqrt{1-x^2} \right) + C.} \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right) &= \frac{1}{2} \left(\frac{1-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right) \\ &= \frac{1}{2} \left(\sqrt{1-x^2} + \sqrt{1-x^2} \right) \checkmark \end{aligned}$$

7. **Find** $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$.

The top and the bottom both go to zero, so L'hospital's rule shows that the limit is

$$\lim_{x \rightarrow 0} \frac{-\sin x + x}{4x^3}.$$

The top and the bottom both go to zero, so L'hospital's rule shows that the limit is

$$\lim_{x \rightarrow 0} \frac{-\cos x + 1}{12x^2}.$$

The top and the bottom both go to zero, so L'hospital's rule shows that the limit is

$$\lim_{x \rightarrow 0} \frac{\sin x}{24x}.$$

The top and the bottom both go to zero, so L'hospital's rule shows that the limit is

$$\lim_{x \rightarrow 0} \frac{\cos x}{24} = \boxed{\frac{1}{24}}.$$

8. **Find** $\lim_{x \rightarrow 0} \frac{\cos x}{x-2}$.

The top goes to 1 and the bottom goes to -2 . The limit is $\boxed{\frac{1}{-2}}$.

9. Find the limit of the sequence whose n^{th} term is $a_n = \left(\frac{n-1}{n+1}\right)^n$.

We must find

$$\lim_{n \rightarrow \infty} \left(\frac{n-1}{n+1}\right)^n.$$

This limit has the indeterminate form that the base goes to 1 and the exponent goes to ∞ . Let $y = \left(\frac{n-1}{n+1}\right)^n$. We must find $\lim_{n \rightarrow \infty} y$. We do find

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} n \ln \left(\frac{n-1}{n+1}\right) = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n-1}{n+1}\right)}{\frac{1}{n}}.$$

The top and the bottom both go to 0. Apply L'hospital's rule to get that

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{\frac{1}{n-1} - \frac{1}{n+1}}{\frac{-1}{n^2}} = \lim_{n \rightarrow \infty} \frac{-2n^2}{n^2 - 1} = -2.$$

So, the answer is

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} e^{\ln y} = \boxed{e^{-2}}.$$

One could also do the problem by observing that

$$\lim_{n \rightarrow \infty} \left(\frac{n-1}{n+1}\right)^n = \lim_{n \rightarrow \infty} \left(\frac{(n+1)-2}{n+1}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-2}{n+1}\right)^n.$$

Let $m = n + 1$. This limit is equal to

$$\lim_{m \rightarrow \infty} \left(1 + \frac{-2}{m}\right)^{m-1} = \lim_{m \rightarrow \infty} \frac{\left(1 + \frac{-2}{m}\right)^m}{1 + \frac{-2}{m}}.$$

We know that

$$\lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^m = e^r;$$

so, the limit of the top is e^{-2} and the limit of the bottom is 1. Once again, we get the answer e^{-2} .

10. Find $\int_{-1}^3 \frac{1}{x^2} dx$.

The function $f(x) = \frac{1}{x^2}$ goes to infinity as x approaches zero. For all x other than zero, $f(x)$ is positive. Thus, this integral represents an area. Either this integral is infinite; or else, the integral is finite and positive.

$$\begin{aligned} \int_{-1}^3 \frac{1}{x^2} dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{x^2} dx + \lim_{a \rightarrow 0^+} \int_a^3 \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow 0^-} \frac{-1}{x} \Big|_{-1}^b + \lim_{a \rightarrow 0^+} \frac{-1}{x} \Big|_a^3 = \lim_{b \rightarrow 0^-} \left(\frac{-1}{b} - \frac{-1}{-1}\right) + \lim_{a \rightarrow 0^+} \left(\frac{-1}{3} - \frac{-1}{a}\right). \end{aligned}$$

Notice that

$$\lim_{b \rightarrow 0^-} \left(\frac{-1}{b}\right) = +\infty \quad \text{and} \quad \lim_{a \rightarrow 0^+} \left(-\frac{1}{a}\right) = +\infty.$$

Thus, the integral diverges to $+\infty$. Notice that $-\frac{4}{3}$ has NOTHING TO DO WITH THE FINAL ANSWER. An "answer" $-\frac{4}{3}$ will receive a score of 0.