PRINT Your Name:__________________

Quiz 8 — March 2, 2012 – Section 7 – 11:15 – 12:05

Remove everything from your desk except a pencil or pen.

Write in complete sentences.
The quiz is worth 5 points.

Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = 3 - \frac{1}{a_n}$.

1. Prove that $1 \le a_n \le a_{n+1}$ for all positive integers n.
2. Prove that $a_n < 3$ for all positive integers n.
3. State the Completeness Axiom and draw a conclusion about the sequence $\{a_n\}$ from the Completeness Axiom.
4. Find the limit of the sequence $\{a_n\}$.

Answer:

(1) We use the technique of Mathematical Induction. We see that $a_1 = 1$ and $a_2 = 2$; so $1 \le a_1 \le a_2$. Assume **BY INDUCTION** that $1 \le a_{n-1} \le a_n$ for some **FIXED** n. Divide both sides of $a_{n-1} \le a_n$ by the **positive** number $a_{n-1}a_n$ to see that $\frac{1}{a_n} \le \frac{1}{a_{n-1}}$. Add $-\frac{1}{a_n} - \frac{1}{a_{n-1}}$ to both sides to see that $-\frac{1}{a_{n-1}} < -\frac{1}{a_n}$. Add 3 to both sides; obtain $3 - \frac{1}{a_{n-1}} \le 3 - \frac{1}{a_n}$; that is, $a_n \le a_{n+1}$. We already had $1 \le a_n$; so indeed,

$$1 \le a_{n-1} \le a_n \implies 1 \le a_n \le a_{n+1}.$$

We saw that $1 \le a_n \le a_{n+1}$ for $n = 1$. We proved that if $1 \le a_{n-1} \le a_n$ for some FIXED n, then $1 \le a_n \le a_{n+1}$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $1 \le a_n \le a_{n+1}$ for ALL positive integers n.

(2) We use the technique of Mathematical Induction. We see that $a_1 = 1$ and $1 < 3$. Assume **BY INDUCTION** that $a_{n-1} < 3$ for some **FIXED** n. We showed in (1) that a_{n-1} is positive. Divide both sides of $a_{n-1} < 3$ by the **positive** number $3a_{n-1}$ to see that $\frac{1}{3} \le \frac{1}{a_{n-1}}$. Add $-\frac{1}{3} - \frac{1}{a_{n-1}}$ to both sides to see that $-\frac{1}{a_{n-1}} < -\frac{1}{3}$. Add 3 to both sides; obtain $3 - \frac{1}{a_{n-1}} \le 3 - \frac{1}{3} < 3$; that is, $a_n < 3$. We saw that $a_n < 3$ for $n = 1$. We proved that if $a_{n-1} < 3$ for some FIXED n, then $a_n < 3$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $a_n < 3$ for ALL positive integers n.

(3) The completeness axiom says that every increasing bounded sequence of real numbers has a limit. We showed in (1) and (2) that $\{a_n\}$ is an increasing bounded sequence of real numbers. We conclude that $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n$.

(4) Take $\lim\limits_{n\to\infty}$ of both sides of $a_{n+1} = 3 - \frac{1}{a_n}$ to conclude that

$$\lim_{n\to\infty} a_{n+1} = 3 - \frac{1}{\lim\limits_{n\to\infty} a_n};$$

that is $L = 3 - \frac{1}{L}$; so $L^2 = 3L - 1$ or $L^2 - 3L + 1 = 0$. The quadratic formula gives $L = \frac{3 \pm \sqrt{9-4}}{2}$. We know that L can not be less than 1 because every term in the sequence is at least 1. So $L \neq \frac{3-\sqrt{5}}{2}$ and hence L does equal $\frac{3+\sqrt{5}}{2}$.