Quiz 8 — October 13, 2010 – Section 10 – 11:15 – 12:05

Consider the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.

(a) Let $M \geq 2$ be some fixed integer. Find a closed formula for the partial sum $s_M = \sum_{n=2}^{M} \frac{2}{n^2-1}$. (Comment. It is possible to use the technique of partial fractions to express this series as a telescoping series.)

(b) What is the sum of the series?

Answer. Write $\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$. Multiply by n^2-1 to see that

$$2 = A(n+1) + B(n-1).$$

Plug in $n=1$ to see that $1 = A$. Plug in $n=-1$ to see that $B=-1$. This works becasue

$$\frac{1}{n-1} + \frac{-1}{n+1} = \frac{n+1-(n-1)}{n^2-1} = \frac{2}{n^2-1}.$$

(a) We see that

$$s_M = \sum_{n=2}^{M} \frac{2}{n^2-1} = \sum_{n=2}^{M} \left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= \begin{cases} \left[\frac{1}{2-1} - \frac{1}{2+1}\right] + \left[\frac{1}{3-1} - \frac{1}{3+1}\right] + \left[\frac{1}{4-1} - \frac{1}{4+1}\right] + \left[\frac{1}{5-1} - \frac{1}{5+1}\right] + \dots \\ \dots + \left[\frac{1}{(M-2)-1} - \frac{1}{(M-2)+1}\right] + \left[\frac{1}{(M-1)-1} - \frac{1}{(M-1)+1}\right] + \left[\frac{1}{M-1} - \frac{1}{M+1}\right] \end{cases}$$

$$= \begin{cases} \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{3} - \frac{1}{5}\right] + \left[\frac{1}{4} - \frac{1}{6}\right] + \dots \\ \dots + \left[\frac{1}{M-3} - \frac{1}{M-1}\right] + \left[\frac{1}{M-2} - \frac{1}{M}\right] + \left[\frac{1}{M-1} - \frac{1}{M+1}\right] \end{cases}$$

$$= \begin{cases} \left[\frac{1}{1} \qquad\right] + \left[\frac{1}{2} \qquad\right] + \left[\qquad\right] + \left[\qquad\right] + \dots \\ \dots + \left[\qquad\qquad\right] + \left[\qquad - \frac{1}{M}\right] + \left[\qquad - \frac{1}{M+1}\right] \end{cases} = \boxed{\frac{3}{2} - \frac{1}{M} - \frac{1}{M+1}.}$$

(b) The sum of the series is

$$\lim_{M\to\infty} s_M = \lim_{M\to\infty} \frac{3}{2} - \frac{1}{M} - \frac{1}{M+1} = \boxed{\frac{3}{2}}.$$