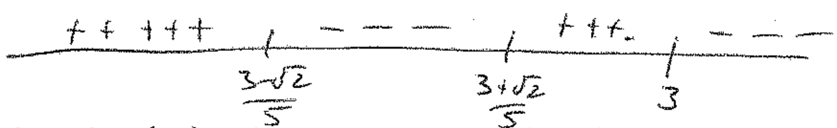
Quiz 7 — March 2, 2011 – Section 4 – 9:05-9:55 recitation.

Remove everything from your desk except this page and a pencil or pen.

The quiz is worth 5 points.

Show that the sequence defined by $a_1 = 2$, $a_{n+1} = \frac{1}{3-a_n}$ satisfies $0 < a_n \le 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Answer. We first show that the sequence $\{a_n\}$ is decreasing. We must show that $a_{n+1} \le a_n$. We must show that $\frac{1}{3-a_n} \le a_n$. We must show that $-a_n + \frac{1}{3-a_n} \le 0$. We must show that $\frac{a_n^2-3a_n+1}{3-a_n} \le 0$. The roots of the numerator are $a_n = \frac{3\pm\sqrt{5}}{2}$. We see that the sign of the expression $\frac{a_n^2-3a_n+1}{3-a_n}$ is given by:



to show that $\{a_n\}$ is decreasing, we must show that

$$(*)\qquad \frac{3-\sqrt{5}}{2} \le a_n \le \frac{3+\sqrt{5}}{2}$$

for all n. (Of course, $\frac{3-\sqrt{5}}{2}$ is a little more than 0 and $\frac{3+\sqrt{5}}{2}$ is a little less than 3.) It certainly is true that $\frac{3-\sqrt{5}}{2} \le a_1 \le \frac{3+\sqrt{5}}{2}$. Suppose that $\frac{3-\sqrt{5}}{2} \le a_{n-1} \le \frac{3+\sqrt{5}}{2}$, then

$$-\frac{3+\sqrt{5}}{2} \le -a_{n-1} \le -\frac{3-\sqrt{5}}{2}$$

and

$$3 - \frac{3+\sqrt{5}}{2} \le 3 - a_{n-1} \le 3 - \frac{3-\sqrt{5}}{2}.$$

We see that $3 - \frac{3+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$ and $3 - \frac{3-\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$. We have

$$\frac{3-\sqrt{5}}{2} \le 3 - a_{n-1} \le \frac{3+\sqrt{5}}{2}.$$

All of these numbers are positive; so,

$$\frac{2}{3+\sqrt{5}} \le \frac{1}{3-a_{n-1}} \le \frac{2}{3-\sqrt{5}}.$$

We see that $\frac{1}{3-a_{n-1}} = a_n$. We calculate

$$\frac{2}{3-\sqrt{5}} = \frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{2(3+\sqrt{5})}{4} = \frac{3+\sqrt{5}}{2}$$

and

$$\frac{2}{3+\sqrt{5}} = \frac{2(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{2(3-\sqrt{5})}{4} = \frac{3-\sqrt{5}}{2}.$$

If (*) holds at $n-1$, then we have shown that

$$\frac{3-\sqrt{5}}{2} \le a_n \le \frac{3+\sqrt{5}}{2}.$$

In other words

$$(*) \text{ holds at } n-1 \implies (*) \text{ holds at } n.$$

We also know that (*) holds at 1. It follows that (*) holds at all n. We have shown that $\{a_n\}$ is a decreasing sequence. We have $a_1 = 2$ and $\{a_n\}$ is a decreasing sequence; thus, $a_n \le 2$ for all n.

The completeness axiom yields that $\{a_n\}$ converges. Let $L = \lim\limits_{n\to\infty} a_n$. Take $\lim\limits_{n\to\infty}$ of both sides of $a_{n+1} = \frac{1}{3-a_n}$ to obtain $\lim\limits_{n\to\infty} a_{n+1} = \frac{1}{3-\lim\limits_{n\to\infty} a_n}$ or $L = \frac{1}{3-L}$. Multiply both sides by $3-L$ to obtain $L(3-L) = 1$; so $3L - L^2 = 1$. It follows that $L = \frac{3-\sqrt{5}}{2}$ or $L = \frac{3+\sqrt{5}}{2}$. We know that $L \ne \frac{3+\sqrt{5}}{2}$ because $\frac{3+\sqrt{5}}{2} > 2 = a_1 \ge a_2 \ge a_3 \dots$. Thus, L must equal $\frac{3-\sqrt{5}}{2}$.