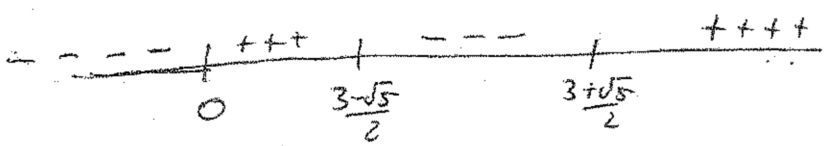
Quiz 7 — March 2, 2011 – Section 3 – 8:00-8:50 recitation.

Remove everything from your desk except this page and a pencil or pen.

The quiz is worth 5 points.

Show that the sequence defined by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and $a_n < 3$ for all n. Deduce that the sequence is convergent and find its limit.

Answer. We need to show that $a_n \le a_{n+1}$. We need to show that $a_n \le 3 - \frac{1}{a_n}$. We need to show that $a_n - 3 + \frac{1}{a_n} \le 0$. We need to show that $\frac{a_n^2 - 3a_n + 1}{a_n} \le 0$. The numerator has roots $a_n = \frac{3 \pm \sqrt{9-4}}{2}$. So the expression $\frac{a_n^2-3a_n+1}{a_n}$ takes the following signs



We must show that

$$(*) \qquad \frac{3-\sqrt{5}}{2} \le a_n \le \frac{3+\sqrt{5}}{2}$$

for all n. (Of course, $\frac{3-\sqrt{5}}{2}$ is a little more than 0 and $\frac{3+\sqrt{5}}{2}$ is a little less than 3.) It certainly is true that $\frac{3-\sqrt{5}}{2} \le a_1 \le \frac{3+\sqrt{5}}{2}$. Suppose that $\frac{3-\sqrt{5}}{2} \le a_{n-1} \le \frac{3+\sqrt{5}}{2}$, then

$$\frac{2}{3+\sqrt{5}} \le \frac{1}{a_{n-1}} \le \frac{2}{3-\sqrt{5}}$$

and

$$-\frac{2}{3-\sqrt{5}} \le -\frac{1}{a_{n-1}} \le -\frac{2}{3+\sqrt{5}}$$

and

$$3 - \frac{2}{3-\sqrt{5}} \le 3 - \frac{1}{a_{n-1}} \le 3 - \frac{2}{3+\sqrt{5}}.$$

But $3 - \frac{1}{a_{n-1}} = a_n$,

$$3 - \frac{2}{3+\sqrt{5}} = 3 - \frac{2(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = 3 - \frac{2(3-\sqrt{5})}{4} = 3 - \frac{(3-\sqrt{5})}{2} = \frac{(3+\sqrt{5})}{2},$$

and

$$3 - \frac{2}{3-\sqrt{5}} = 3 - \frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = 3 - \frac{2(3+\sqrt{5})}{4} = 3 - \frac{(3+\sqrt{5})}{2} = \frac{3-\sqrt{5}}{2}.$$

We have shown that (*) holds at $n = 1$ and if (*) holds at $n - 1$, then (*) holds at n. We conclude that (*) holds at all n.

In particular, we have shown that the sequence $\{a_n\}$ is increasing and that $a_n < 3$ for all n.

The completeness axiom yields that $\{a_n\}$ converges. Let $L = \lim_{n\to\infty} a_n$. Take $\lim_{n\to\infty}$ of both sides of $a_{n+1} = 3 - \frac{1}{a_n}$ to obtain $\lim_{n\to\infty} a_{n+1} = 3 - \frac{1}{\lim_{n\to\infty} a_n}$ or $L = 3 - \frac{1}{L}$. Multiply both sides by L to obtain $L^2 = 3L - 1$; so $L = \frac{3-\sqrt{5}}{2}$ or $L = \frac{3+\sqrt{5}}{2}$. We know that $L \neq \frac{3-\sqrt{5}}{2}$ because $\frac{3-\sqrt{5}}{2} < 1 = a_1 \leq a_2 \leq a_3 \ldots$. Thus, L must equal $\frac{3+\sqrt{5}}{2}$.