PRINT Your Name:____________________

Quiz 11 — September 22, 2015

Remove everything from your desk except this page and a pencil or pen. The solution will be posted soon after the quiz is given.

Circle your answer. **Show your work. Your work must be correct and coherent. Check your answer.**

Find $\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx$.

Answer: We see that $x^2-4x+6=(x-2)^2+2$, which is an irreducible quadratic. Let $u=x-2$. It follows that $du=dx$. It also follows that

$$x^2-3x+7=(u+2)^2-3(u+2)+7=u^2+u+5.$$

So,

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx=\int \frac{u^2+u+5}{(u^2+2)^2}du.$$

We use the technique of partial fractions. We look for constants A, B, C, and D with

$$\frac{u^2+u+5}{(u^2+2)^2}=\frac{Au+B}{u^2+2}+\frac{Cu+D}{(u^2+2)^2}.$$

Multiply both sides by $(u^2+2)^2$ to obtain

$$u^2+u+5=(Au+B)(u^2+2)+Cu+D.$$

So,

$$u^2+u+5=Au^3+Bu^2+(2A+C)u+(2B+D).$$

Equate the corresponding coefficients.

$$0=A,\quad 1=B,\quad 1=2A+C,\quad 5=2B+D.$$

Thus,

$$0=A,\quad 1=B,\quad 1=C,\quad 3=D.$$

We claim that

$$\frac{u^2+u+5}{(u^2+2)^2}=\frac{1}{u^2+2}+\frac{u+3}{(u^2+2)^2}.$$

Of course the right side is

$$\frac{u^2+2+u+3}{(u^2+2)^2},$$

which is the left side. We see that

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx = \int \frac{u^2+u+5}{(u^2+2)^2}du = \int \frac{1}{u^2+2} + \frac{u+3}{(u^2+2)^2}du$$

$$= \frac{1}{2}\int \frac{1}{(\frac{u}{\sqrt{2}})^2+1}du - \frac{1}{2(u^2+2)} + \frac{3}{4}\int \frac{1}{((\frac{u}{\sqrt{2}})^2+1)^2}du.$$

Let $w = \frac{u}{\sqrt{2}}$. It follows that $dw = \frac{du}{\sqrt{2}}$. The integral is

$$= \frac{\sqrt{2}}{2}\int \frac{1}{w^2+1}dw - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{4}\int \frac{1}{(w^2+1)^2}dw.$$

Let $w = \tan\theta$. Compute $w^2+1 = \tan^2\theta + 1 = \sec^2\theta$ and $dw = \sec^2\theta d\theta$. The integral is

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{4}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{4}\int \cos^2\theta d\theta$$

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{8}\int (1+\cos 2\theta)d\theta$$

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{\sqrt{2}}{2}\arctan w + \frac{1}{2(u^2+2)} - \frac{3\sqrt{2}}{8}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C$$

$$= \frac{\sqrt{2}}{2}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C.$$

At this point you might want to draw a right triangle with $\tan\theta = w$. So make w be the length of the side opposite θ and 1 be the length of the side adjacent to θ. Calculate that the hypotenuse has length $\sqrt{w^2+1}$. It follows that $\sin\theta = \frac{w}{\sqrt{w^2+1}}$ and $\cos\theta = \frac{1}{\sqrt{w^2+1}}$. Of course $\theta = \arctan w$. We add the two terms that look like a constant times $\arctan w$. At any rate, the integral is

$$= \frac{7\sqrt{2}}{8}\arctan w - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{8}\frac{w}{w^2+1} + C$$

$$= \frac{7\sqrt{2}}{8} \arctan\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2(u^2+2)} + \frac{3\sqrt{2}}{8} \frac{\frac{u}{\sqrt{2}}}{(\frac{u}{\sqrt{2}})^2+1} + C$$

$$= \frac{7\sqrt{2}}{8} \arctan\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2(u^2+2)} + \frac{3}{4} \frac{u}{u^2+2} + C$$

$$= \frac{7\sqrt{2}}{8} \arctan\left(\frac{u}{\sqrt{2}}\right) + \frac{-2+3u}{4(u^2+2)} + C$$

$$= \frac{7\sqrt{2}}{8} \arctan\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-2+3(x-2)}{4((x-2)^2+2)} + C$$

$$= \boxed{\frac{7\sqrt{2}}{8} \arctan\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C}$$

Check: The derivative of the proposed answer is

$$\frac{7\sqrt{2}}{8} \frac{1}{\sqrt{2}((\frac{x-2}{\sqrt{2}})^2+1)} + \frac{3(x^2-4x+6)-(3x-8)(2x-4)}{4(x^2-4x+6)^2}$$

$$= \frac{7}{8} \frac{1}{((\frac{x-2}{\sqrt{2}})^2+1)} + \frac{-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{7}{4} \frac{1}{((x-2)^2+2)} + \frac{-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{7}{4(x^2-4x+6)} + \frac{-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{7(x^2-4x+6)}{4(x^2-4x+6)^2} + \frac{-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{7(x^2-4x+6)-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{7x^2-28x+42-3x^2+16x-14}{4(x^2-4x+6)^2}$$

$$= \frac{4x^2-12x+28}{4(x^2-4x+6)^2} = \frac{4(x^2-3x+7)}{4(x^2-4x+6)^2} = \frac{x^2-3x+7}{(x^2-4x+6)^2}. \checkmark$$