Final Exam, Math 142, Fall 1998, Problems 5 through 15

5. Find $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\,dx$.

The function $f(x) = \frac{1}{x^2}$ goes to plus infinity as x goes to zero. This integral is improper.

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx + \lim_{a\to 0^+} \int_{a}^{2} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to 0^-} -\frac{1}{x}\Big|_{-1}^{b} + \lim_{a\to 0^+} -\frac{1}{x}\Big|_{a}^{2}$$

$$= \lim_{b\to 0^-} -\frac{1}{b} - 1 + \lim_{a\to 0^+} -\frac{1}{2} + \frac{1}{a} = -\frac{3}{2} + \infty + \infty = \boxed{+\infty}.$$

6. Find $\displaystyle\int xe^{3x}\,dx$.

Use integration by parts with

$$\begin{aligned} u &= x & v &= \tfrac{1}{3}e^{3x} \\ du &= dx & dv &= e^{3x}dx. \end{aligned}$$

Thus,

$$\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.}$$

7. Find $\displaystyle\int \frac{5x^2 - x - 1}{x^3 - x^2}\,dx$.

Start with

$$\frac{5x^2 - x - 1}{x^3 - x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiply both sides by $x^3 - x^2$ to get

$$\begin{aligned} 5x^2 - x - 1 =& Ax(x-1) + B(x-1) + Cx^2, \\ 5x^2 - x - 1 =& (A + C)x^2 + (B - A)x - B. \end{aligned}$$

Equate the corresponding coefficients:

$$\begin{aligned} 5 =& A + C \\ -1 =& B - A \\ -1 =& -B. \end{aligned}$$

We see that $B = 1$, $A = 2$, and $C = 3$. Thus,

$$\int \frac{5x^2 - x - 1}{x^3 - x^2}\, dx = \int \frac{2}{x} + \frac{1}{x^2} + \frac{3}{x-1}\, dx = \boxed{2\ln|x| - \frac{1}{x} + 3\ln|x-1| + C}.$$

8. Find $\int \sin^6 x \cos^3 x\, dx$.

Let $u = \sin x$. It follows that $du = \cos x\, dx$ and

$$\int \sin^6 x \cos^3 x\, dx = \int \sin^6 x(1 - \sin^2 x)\cos x\, dx = \int (u^6 - u^8)du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C = \boxed{\frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C}.$$

9. Find $\int \frac{1}{\sqrt{x^2 + 4x + 5}}\, dx$.

We see that

$$\int \frac{1}{\sqrt{x^2 + 4x + 5}}\, dx \int \frac{1}{\sqrt{(x+2)^2 + 1}}\, dx.$$

We make a trig substitution. Let $x + 2 = \tan\theta$. Then, the following equations all hold:

$$dx = \sec^2\theta\, d\theta, \quad (x+2)^2 + 1 = \tan^2\theta + 1 = \sec^2\theta.$$

We now see that

$$\int \frac{1}{\sqrt{x^2 + 4x + 5}}\, dx = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln|\sqrt{x^2 + 4x + 5} + x + 2| + C}.$$

10. Find $\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}}$.

Apply L'Hopital's rule four times. Each time the top and the bottom each go to zero.

$$\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}} = \lim_{x\to 0} \frac{4x^3}{-\sin x + x} = \lim_{x\to 0} \frac{12x^2}{-\cos x + 1} = \lim_{x\to 0} \frac{24x}{\sin x}$$

$$= \lim_{x\to 0} \frac{24}{\cos x} = \boxed{24}.$$

11. Find $\lim_{x\to\infty} \left(1 - \frac{1}{2x}\right)^x$.

Let $y = \left(1 - \frac{1}{2x}\right)^x$**. We see that**

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 - \frac{1}{2x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{1}{2x}\right)}{\frac{1}{x}}.$$

The top and the bottom both go to zero, so we may apply L'Hopital's rule to see that

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{-1}{2} \cdot \frac{-1}{x^2} \cdot \frac{1}{1-\frac{1}{2x}}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{-1}{2} \cdot \frac{1}{1 - \frac{1}{2x}} = \frac{-1}{2}.$$

We now see that

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = \boxed{e^{-\frac{1}{2}}}.$$

12. Where does the power series function $f(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n}$ converge?

We see that $f(x)$ **converges for** $x = 5$**; henceforth, we study** $x \neq 5$**. Let**

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-5)^{n+1}}{2^{n+1}}}{\frac{(x-5)^n}{2^n}}\right| = \lim_{n\to\infty} \frac{|x-5|}{2} = \frac{|x-5|}{2}.$$

If $\rho < 1$**, then the series converges. If** $1 < \rho$**, then the series diverges. We see that** $\rho < 1$ **precisely, when** $-1 < \frac{x-5}{2} < 1$**; that is,** $-2 < x - 5 < 2$**;** $3 < x < 7$**. We also see that** $1 < \rho$ **when** $x < 3$**; or else,** $7 < x$**. We need only consider** $x = 3$ **and** $x = 7$**. We see that**

$$f(3) = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n.$$

The series $\sum_{n=1}^{\infty} (-1)^n$ **diverges by the** n^{th} **term test. We see that**

$$f(7) = \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1.$$

The series $\sum_{n=1}^{\infty} 1$ **diverges by the** n^{th} **term test. We conclude that**

$$\boxed{f(x) \textbf{ converges for } 3 < x < 7\textbf{, and diverges everywhere else.}}$$

13. What familiar function is equal to $f(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \ldots$?

We know that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

It follows that

$$\boxed{e^{x^2}} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \dots.$$

14. Does $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{(3)^{\frac{1}{n}}}$ converge? (Explain your answer.)

Observe that $\lim\limits_{n\to\infty} \frac{1}{(3)^{\frac{1}{n}}} = 1$. **The series** $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{(3)^{\frac{1}{n}}}$ **DIVERGES by the** n^{th} **term test.**

15. Does $\displaystyle\sum_{n=2}^{\infty}\frac{\sin n}{n^3}$ converge? (Explain your answer.)

The series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3}$ **is a** p**-series with** $p = 3 > 1$; **so,** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3}$ **converges. We know that** $\frac{|\sin n|}{n^3} < \frac{1}{n^3}$; **so the comparison test yields that** $\displaystyle\sum_{n=2}^{\infty}\frac{|\sin n|}{n^3}$ **converges. Finally, the Absolute convergence test yields that**

$$\boxed{\sum_{n=2}^{\infty}\frac{\sin n}{n^3} \textbf{ CONVERGES.}}$$