Math 142, Exam 2, Fall, 2004

PRINT Your Name:______________________________________

There are 11 problems on 5 pages. The exam is worth 100 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office door tomorrow morning, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **(9 points) Solve** $3\log_2 x = 7 + \log_2(2x)$ **. Check your answer.**

Exponentiate to get

$$(2^{\log_2 x})^3 = 2^7 2^{\log_2(2x)}.$$

We must solve $x^3 = 2^7 2x$. We must solve $x^3 - 2^8 x = 0$. Factor to get $x(x-2^4)(x+2^4) = 0$. Thus, $x = 0$, or $x = -16$, or $x = 16$. Neither $x = 0$ nor $x = -16$ are in the domain of $\log_2 x$. The only possible solution is $\boxed{x = 16}$. This works because when we plug $x = 16$ into the original problem the left side becomes $3\log_2 16 = 3(4) = 12$ and the right side becomes $7 + \log_2(32) = 7 + 5 = 12$.

2. **(9 points) Find the area of the region bounded by** $y = \frac{\ln x}{x}$ **, the** x **-axis,** $x = e^2$ **and** $x = e^3$ **.**

The function $\frac{\ln x}{x}$ is positive for the x 's under considerastion; so the area is the integral from the left side to the right side of the top minus the bottom:

$$\int_{e^2}^{e^3} \frac{\ln x}{x} dx = \int_2^3 u du = \frac{u^2}{2}\bigg|_2^3 = \frac{9}{2} - \frac{4}{2} = \boxed{\frac{5}{2}}.$$

Let $u = \ln x$; so $du = \frac{1}{x} dx$. When $x = e^2$, $u = 2$ and when $x = e^3$, then $u = 3$.

3. **(10 points) If** $y = 3x^2 \arcsin(2x)$ **, then find** $\frac{dy}{dx}$ **.**

$$\frac{dy}{dx} = \boxed{\frac{3x^2 \cdot 2}{\sqrt{1-4x^2}} + 6x\arcsin(2x).}$$

4. **(9 points) The population of the United States was 3.9 million in 1790 and 178 million in 1960. If the rate of growth is assumed proportional to the number present, what estimate would you give for the population in 2000? (You may leave** $\ln$ **in your answer.)**

Let $P(t)$ equal the population at time t , where t is measured in tears, with $t = 0$ taken to be 1790. We are told that $P(0) = 3.9 \times 10^6$ and $P(170) = 178 \times 10^6$.

We want to find $P(210)$. The words which come between "If" and the comma tell us that $\frac{dP}{dt} = kP$ for some constant k. We solved this differential equation in class and learned that $P(t) = P(0)e^{kt}$. Plug in $t = 170$ to see that $178 \times 10^6 = 3.9 \times 10^6 e^{170k}$. In other words, $\frac{178}{3.9} = e^{170k}$ and $\ln\left(\frac{178}{3.9}\right) = 170k$; hence, $\frac{1}{170}\ln\left(\frac{178}{3.9}\right) = k$. Now we know that

$$P(t) = 3.9(10^6)e^{\frac{t}{170}\ln\left(\frac{178}{3.9}\right)}$$

for all times t. In particular,

$$\boxed{P(210) = 3.9(10^6)e^{\frac{210}{170}\ln\left(\frac{178}{3.9}\right)}.}$$

If you are sitting at home with a calculator in your hand, you compute

$$3.9(10^6)e^{\frac{210}{170}\ln\left(\frac{178}{3.9}\right)} \approx 437.3 \times 10^6.$$

We notice that this number is significantly higher than the actual US population in 2000. Can you think of any cultural happenings that occurred about 1960 which caused a big change in the birth rate in the United States?

5. **(9 points) Find the value of** $\sin(2\arccos\frac{5}{13})$**.**

$$\sin(2\arccos\frac{5}{13}) = 2\sin(\arccos\frac{5}{13})\cos(\arccos\frac{5}{13}) = 2\frac{12}{13}\frac{5}{13} = \boxed{\frac{120}{169}}.$$

6. **(9 points) Find** $\int \sin^5 x\cos^2 x\,dx$**. Check your answer.**

$$\int \sin^5 x\cos^2 x\,dx = \int (1-cos^2 x)^2\cos^2 x\sin x\,dx = -\int (1-u^2)^2u^2\,du$$

$$= -\int (u^2 - 2u^4 + u^6)\,du = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= \boxed{-\left(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7}\right) + C.}$$

7. **(9 points) Find** $\int \frac{1}{\sqrt{9-4x^2}}\,dx$**. Check your answer.**

Let $2x = 3\sin\theta$. It follows that $2dx = 3\cos\theta\,d\theta$ and that $\sqrt{9-4x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$. The integral is equal to

$$\frac{1}{2}\int \frac{3\cos\theta\,d\theta}{3\cos\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \boxed{\frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C.}$$

8. **(9 points) Find** $\int \frac{x}{\sqrt{9-4x^2}}\,dx$ **. Check your answer.**

Let $u = 9 - 4x^2$. Observe that $du = -8x\,dx$. The integral is

$$\frac{-1}{8}\int u^{-1/2}\,du = \frac{-\sqrt{u}}{4} + C = \boxed{\frac{-\sqrt{9-4x^2}}{4} + C}.$$

9. **(9 points) Find** $\int \frac{1}{\sqrt{9+4x^2}}\,dx$ **. Check your answer.**

Let $2x = 3\tan\theta$. Observe that $2dx = 3\sec^2\theta\,d\theta$. Observe also that $\sqrt{9+4x^2} = \sqrt{9+9\tan^2\theta} = \sqrt{9\sec^2\theta} = 3\sec\theta$. The integral is equal to

$$\frac{1}{2}\int \frac{3\sec^2\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C}$$

10. **(9 points) Find** $\int \cos^4 x\,dx$ **. Check your answer.**

Use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. The integral is

$$\int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2\,dx = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \tfrac{1}{4}\int (1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x))\,dx$$

$$= \tfrac{1}{4}\int (\tfrac{3}{2} + 2\cos 2x + \tfrac{1}{2}\cos 4x)\,dx$$

$$\boxed{= \tfrac{1}{4}(\tfrac{3}{2}x + \sin 2x + \tfrac{1}{8}\sin 4x) + C.}$$

11. **(9 points) Find** $\int \sin 2x \sin 3x\,dx$ **. Check your answer.**

Subtract the lower identity from the higher identity

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

to get:

$$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$$

or

$$\frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) = \sin A\sin B.$$

Let $A = 2x$ and $B = 3x$ to get

$$\int \sin 2x\sin 3x\,dx = \frac{1}{2}\int (\cos(-x) - \cos(5x))\,dx = \frac{1}{2}\int (\cos(x) - \cos(5x))\,dx$$

$$= \boxed{\frac{1}{2}\left(\sin x - \frac{\sin(5x)}{5}\right) + C.}$$