Math 142, Exam 2, Fall 2002

Name ________________________________

There are 10 problems on 6 pages. Each problem is worth 10 points. each. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** CHECK your answer whenever possible.

1. **Find** $\int \sin^5 x\,dx$ **. Check your answer.**

Let $u = \cos x$. It follows that $du = -\sin x dx$. We have

$$\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx = -\int (1-u^2)^2\,du = -\int (1-2u^2+u^4)\,du$$

$$= -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C = \boxed{-\left(\cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5}\right) + C}$$

Check. The derivative of $-\left(\cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5}\right)$ is

$$-(-\sin x + 2\cos^2 x \sin x - \cos^4 x \sin x) = \sin x(1 - 2\cos^2 x + \cos^4 x)$$

$$= \sin x(1-\cos^2 x)^2 \checkmark.$$

2. **Find** $\int \cos^4 x\,dx$ **.**

We have

$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$$

$$= \boxed{\frac{1}{4}\left[\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right] + C.}$$

3. **Find** $\int \sin 4x \cos 5x\,dx$ **.**

Add

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ and

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ to get

$$\frac{1}{2}[\sin(A+B) + \sin(A-B)] = \sin A \cos B.$$

Take $A = 4x$ and $B = 5x$ to get

$$\int \sin 4x \cos 5x\,dx = \frac{1}{2}\int (\sin 9x - \sin x)\,dx = \boxed{\frac{1}{2}\left(\frac{-\cos 9x}{9} + \cos x\right) + C.}$$

4. **Find** $\int \sec^3 x\,dx$ **. Check your answer.**

Use integration by parts with $u = \sec x$ and $dv = \sec^2 x dx$. It follows that $du = \sec x \tan x dx$ and $v = \tan x$. Thus,

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx.$$

I know $\sin^2 x + \cos^2 x = 1$. So, I also know $\tan^2 x + 1 = \sec^2 x$. So,

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$$

Add $\int \sec^3 x\,dx$ to both sides to see that

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C.$$

We conclude that

$$\boxed{\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + K.}$$

Check. The derivative of $\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)$ is

$$\frac{1}{2}(\sec^3 x + \sec x \tan^2 x + \sec x) = \frac{1}{2}(\sec^3 x + \sec x(\tan^2 x + 1)).\checkmark$$

5. **Find** $\int x \cos x\,dx$ **. Check your answer.**

Use integration by parts. Take $u = x$ and $dv = \cos x$. We calculate $du = dx$ and $v = \sin x$. We have

$$\int x \cos x\,dx = x \sin x - \int \sin x dx = \boxed{x \sin x + \cos x + C.}$$

Check. The derivative of $x \sin x + \cos x$ is

$$x \cos x + \sin x - \sin x.\checkmark$$

6. **Find** $\int \frac{\sqrt{1-x^2}}{x}\,dx$ **.**

Let $x = \sin\theta$. In this case, $\sqrt{1-x^2} = \cos\theta$, $dx = \cos\theta d\theta$, and

$$\int \frac{\sqrt{1-x^2}}{x}\,dx = \int \frac{\cos^2\theta}{\sin\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \int \csc\theta - \sin\theta\,d\theta$$

$$= \int \frac{\csc\theta(\csc\theta+\cot\theta)}{\csc\theta+\cot\theta} - \sin\theta\, d\theta = -\ln|\csc\theta+\cot\theta| + \cos\theta + C$$

$$= -\ln\left|\frac{1+\cos\theta}{\sin\theta}\right| + \cos\theta + C = -\ln|1+\cos\theta| + \ln|\sin\theta| + \cos\theta + C$$

$$= \boxed{-\ln|1+\sqrt{1-x^2}| + \ln|x| + \sqrt{1-x^2} + C.}$$

Check. Observe that the derivative of $-\ln(1+\sqrt{1-x^2}) + \ln x + \sqrt{1-x^2}$ is

$$\frac{\frac{x}{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}} + \frac{1}{x} + \frac{-x}{\sqrt{1-x^2}} = \frac{x}{(1+\sqrt{1-x^2})\sqrt{1-x^2}} + \frac{1}{x} + \frac{-x(1+\sqrt{1-x^2})}{\sqrt{1-x^2}(1+\sqrt{1-x^2})}$$

$$= \frac{x - x - x\sqrt{1-x^2}}{\sqrt{1-x^2}(1+\sqrt{1-x^2})} + \frac{1}{x} = \frac{-x}{(1+\sqrt{1-x^2})} + \frac{1}{x} = \frac{-x^2+1+\sqrt{1-x^2}}{x(1+\sqrt{1-x^2})}$$

$$= \frac{\sqrt{1-x^2}(\sqrt{-x^2+1}+1)}{x(1+\sqrt{1-x^2})} = \frac{\sqrt{1-x^2}}{x}. \checkmark$$

7. **If** $y = \arcsin(2x^2)$ **, then find** $\frac{dy}{dx}$ **.**

$$\frac{dy}{dx} = \boxed{\frac{4x}{\sqrt{1-4x^4}}.}$$

8. **Simplify** $\cos[2\arcsin\left(\frac{1}{3}\right)]$ **.**

$$\cos\left[2\arcsin\left(\frac{1}{3}\right)\right] = \cos^2\left[\arcsin\left(\frac{1}{3}\right)\right] - \sin^2\left[\arcsin\left(\frac{1}{3}\right)\right].$$

It is clear that $\sin(\arcsin\left(\frac{1}{3}\right)) = \frac{1}{3}$. It is not hard to see that

$$\cos\left[\arcsin\left(\frac{1}{3}\right)\right] = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{\sqrt{8}}{3}.$$

We conclude that

$$\cos\left[2\arcsin\left(\frac{1}{3}\right)\right] = \frac{8}{9} - \frac{1}{9} = \boxed{\frac{7}{9}}.$$

9. **Find the solution of the differential equation** $\frac{dy}{dx} - \frac{y}{x} = 3x^3$ **which satisfies** $y(1) = 0$ **. Check your answer.**

This is a first order linear differential equation. It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$. Multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

to get

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 3x^2.$$

Integrate both sides, with respect to x, to get $\frac{y}{x} = x^3 + C$. Plug in the initial condition to get $C = -1$. The solution is $\boxed{y = x^4 - x.}$
Check. Notice that when $x = 1$, we have $y = 0$. Notice also that

$$\frac{dy}{dx} - \frac{y}{x} = 4x^3 - 1 - (x^3 - 1) = 3x^3 \checkmark.$$

10. **Let $f(x) = x \ln x$. What is the domain of $f(x)$? Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

$\boxed{\text{The domain of } f(x) \text{ is all positive } x.}$ The limit as x goes to zero from above of $f(x)$ is zero. One needs l'Hopital's rule to see this because x and $\ln x$ are fighting one another since x wants the answer to be zero and $\ln x$ wants the answer to be $-\infty$. The x wins. If you don't know l'Hopital's rule yet, don't worry I won't hold it against you. $f'(x) = 1 + \ln x$. So $f'(x)$ is positive for $1/e < x$ and $f'(x)$ is negative for $0 < x < 1/e$. So,

$$\boxed{\begin{array}{c} f(x) \text{ is increasing for } 1/e < x \text{ and } f(x) \text{ is decreasing for } 0 < x < 1/e. \\ \text{The point } (1/e, -1/e) \text{ is a local minimum for } y = f(x). \end{array}}$$

We see that $f''(x) = \frac{1}{x}$ which is always positve.

$$\boxed{\begin{array}{c} \text{The graph is always concave up. The graph is never concave down.} \\ \text{There are no points of inflection.} \end{array}}$$

The graph appears on a separate page.