Solutions to Exam 1, Fall 2002, Math 142

1. **Find** $\int \frac{e^x}{\sqrt{1-e^x}}dx$ **.**
   Let $u = 1 - e^x$. It follows that $du = -e^x dx$ and the original problem is

$$-\int u^{-1/2}du = -2\sqrt{u} + c = \boxed{-2\sqrt{1-e^x} + C}.$$

Notice that the derivative of the proposed answer is equal to $\frac{e^x}{\sqrt{1-e^x}}$, as expected.

2. **Find** $\int 2^x + x^2 dx$ **.**

The original problem is equal to

$$\int 2^x + x^2 dx = \int e^{x \ln 2} + x^2 dx = \frac{1}{\ln 2}e^{x \ln 2} + \frac{x^3}{3} + C = \boxed{\frac{1}{\ln 2}2^x + \frac{x^3}{3} + C.}$$

3. **If** $y = e^{\frac{1}{x^2}} + \frac{1}{e^{x^2}}$ **, then find** $\frac{dy}{dx}$ **.**

The original problem is equal to $y = e^{(x^{-2})} + e^{(-x^2)}$. So,

$$\frac{dy}{dx} = \boxed{-2x^{-3}e^{x^{-2}} - 2xe^{-x^2}}.$$

4. **Find the volume of the solid generated by revolving the region bounded by** $y = e^x$ **, the** $x-$ **axis, the** $x = 1$ **and** $x = 2$ **, about the** x **-axis.**

Chop the x -axis from $x = 1$ to $x = 2$ into subintervals. Over each subinterval draw the rectangle from the x -axis up to the curve. Spin each rectangle. Get a disk of volume $\pi r^2 t$, where t , which is the thickness of the disk, is dx and r , which is the radius of the disk, is the y -coordinate at the top of the rectangle, writen in terms of x . In other words $r = e^x$. The volume of the solid is

$$\pi \int_1^2 e^{2x}dx = \left.\frac{\pi e^{2x}}{2}\right|_1^2 = \boxed{\frac{\pi(e^4 - e^2)}{2}}.$$

See a different page for the picture that goes with this solution.

5. **Let** $f(x) = \frac{2x-1}{x+3}$ **for** $x \neq -3$ **. Find** $f^{-1}(x)$ **. If you have time, verify that** $f(f^{-1}(x)) = x$ **for all** x **in the domain of** $f^{-1}(x)$ **, and** $f^{-1}(f(x)) = x$ **for all** x **in the domain of** $f(x)$ **.**

Let $y = f^{-1}(x)$. It follows that $f(y) = x$. We must find y . We have

$$\frac{2y-1}{y+3} = x,$$

$$(2y-1)=x(y+3), \text{ and}$$
$$-3x-1=y(x-2).$$

So, $y=\frac{1+3x}{2-x}$. Therefore,

$$\boxed{f^{-1}(x)=\frac{1+3x}{2-x}, \quad \text{for } x\neq 2.}$$

We verify that if $x\neq 2$, then

$$f(f^{-1}(x))=f\left(\frac{1+3x}{2-x}\right)=\frac{2\frac{1+3x}{2-x}-1}{\frac{1+3x}{2-x}+3}=\frac{2(1+3x)-(2-x)}{(1+3x)+3(2-x)}=\frac{7x}{7}=x.$$

Also, if $x\neq -3$, then

$$f^{-1}(f(x))=f^{-1}(\frac{2x-1}{x+3})=\frac{1+3\left(\frac{2x-1}{x+3}\right)}{2-\left(\frac{2x-1}{x+3}\right)}=\frac{x+3+3(2x-1)}{2(x+3)-(2x-1)}=\frac{7x}{7}=x.$$

6. **If $y=\arcsin(2x^2)$, then find $\frac{dy}{dx}$.**

$$\boxed{\frac{dy}{dx}=\frac{4x}{\sqrt{1-(2x^2)^2}}}.$$

7. **Let $f(x)=\frac{x}{\ln x}$. What is the domain of $f(x)$? Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y=f(x)$. Find all vertical and horizontal asymptotes of $y=f(x)$. Graph $y=f(x)$.**

$\boxed{\text{The domain of } f \text{ is all positive } x, \text{ except } x=1}$. The limit of $f(x)$ as x goes to 1 from above is plus infinity. The limit of $f(x)$ as x goes to 1 from below is negative infinity. At any rate, $\boxed{\text{the line } x=1 \text{ is a vertical asymptote of } y=f(x).}$ The limit of $f(x)$ as x goes to infinity of $f(x)$ is plus infinity. (I have to use l'Hopital's rule for this calculation. If you don't know l'Hopital's rule then you probably can't do that part of the problem yet. I'll teach this rule in section 9.1. Roughly speaking, l'Hopital's rule will tell us that x is more powerful than $\ln x$, so when these two functions fight, x wins.) The limit of $f(x)$ as x goes to zero from above is 0. (I do not need l'Hopital's rule here. Both x and $\frac{1}{\ln x}$ want the limit to be 0.) At any rate, $\boxed{y=f(x) \text{ does not have any horizontal asymptotes.}}$ We see that $f(x)=x(\ln x)^{-1}$. It follows that

$$f'(x)=-x(\ln x)^{-2}(1/x)+(\ln x)^{-1}=(\ln x)^{-2}(\ln x-1).$$

Thus

$$\boxed{\begin{array}{c}f(x) \text{ is increasing for } e<x \text{ and } f(x) \text{ is decreasing for } 0<x<1 \text{ or } 1<x\leq e.\\ \text{The point } (e,e) \text{ is a local minimum.}\\ \text{The graph does not have any local maximum points.}\end{array}}$$

We also see that

$$f''(x) = (\ln x)^{-2}(1/x) - 2(\ln x - 1)(\ln x)^{-3}(1/x) = (\ln x)^{-3}(1/x)(\ln x - 2\ln x + 2).$$

It follows that

$$f''(x) = \frac{2 - \ln x}{x(\ln x)^3}.$$

Thus

$$\boxed{\begin{array}{c} f(x) \text{ is concave down for } 0 < x < 1 \text{ or } e^2 < x; \\ f(x) \text{ is concave up for } 1 < x < e^2. \\ \text{The point } (e^2, \frac{e^2}{2}) \text{ is a point of inflection.} \end{array}}$$

The graph appears on a different page.

8. **Find the general solution of the differential equation** $\frac{dy}{dt} + y = e^{-t}$ **. Check your answer.**

This is a first order linear differential equation. I multiply both sides of the equation by e^t to get $e^t\frac{dy}{dt} + e^t y = 1$. Integrate both sides with respect to t to get $e^t y = t + C$. The solution is $\boxed{y = te^{-t} + Ce^{-t}}$. Notice that

$$\frac{dy}{dt} + y = \left[-te^{-t} + e^{-t} - Ce^{-t}\right] + \left[te^{-t} + Ce^{-t}\right] = e^{-t},$$

as expected.

9. **Simplify** $\sin[2\arccos(\frac{2}{3})]$ **.**

I know that $\sin(2\theta) = 2\sin\theta\cos\theta$. (Of course, if you know $\sin(x+y) = \sin x\cos y + \cos x\sin y$, then take x and y both equal to θ to get the formula for $\sin(2\theta)$.) So,

$$\sin[2\arccos(\tfrac{2}{3})] = 2\sin[\arccos(\tfrac{2}{3})]\cos[\arccos(\tfrac{2}{3})].$$

It is clear that $\cos[\arccos(\frac{2}{3})] = \frac{2}{3}$. Draw a little triangle to see that $\sin[\arccos(\frac{2}{3})] = \sqrt{1 - \frac{4}{9}}$. The answer is

$$2\tfrac{2}{3}\sqrt{1 - \tfrac{4}{9}} = \boxed{\frac{4\sqrt{5}}{9}}.$$

10. **A bacterial population grows at a rate proportional to its size. Initially the population is $10,000$ and after 10 days the population is $20,000$. How long will it take the population to triple? (You may leave "ln" in your answer.)**

Let $A(t)$ equal the amount of bacteria at time t. We are told that $A(0) = 10,000$ and $A(10) = 20,000$. We want to know when $A(t) = 30,000$. The first sentence tells us that $A(t) = A(0)e^{kt}$ for some k. Plug in $t = 10$ to see that $2 = e^{10k}$; thus, $\frac{\ln 2}{10} = k$ and $A(t) = 10,000e^{\frac{t\ln 2}{10}}$. If $A(t) = 30,000$, then $3 = e^{\frac{t\ln 2}{10}}$; so $\ln 3 = \frac{t\ln 2}{10}$ and $t = \boxed{\frac{10\ln 3}{\ln 2} \text{ days}}$. This number is approximately equal to 15.8 days.